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The other dual of MacMahon's theorem on plane partitions ☆

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ABSTRACT

In this paper we introduce a counterpart structure to the shamrocks studied in the paper *A dual of Macmahon's theorem on plane partitions* by M. Ciucu and C. Krattenthaler (2013) [5], which, just like the latter, can be included at the center of a lattice hexagon on the triangular lattice so that the region obtained from the hexagon by removing it has its number of lozenge tilings given by a simple product formula. The new structure, called a fern, consists of an arbitrary number of equilateral triangles of alternating orientations lined up along a lattice line. The shamrock and the fern seem to be the only such connected structures with this property. It would be interesting to understand the reason for this.

1. Introduction

Few results have influenced more the current intense attention devoted to different aspects of tilings and perfect matchings than MacMahon's classical theorem on the number of plane partitions that fit in a given box (see [12, Sect. 495], and [13,1,11,14,9,5] for

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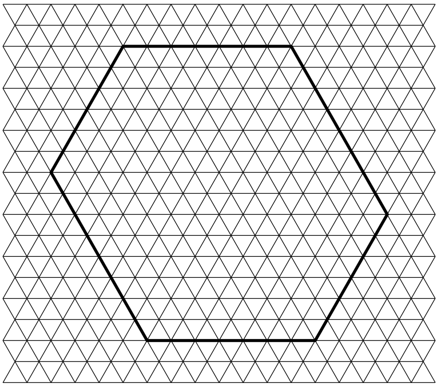

Fig. 1.1. Hexagon with $a = 7$, $b = 6$, $c = 8$.

more recent developments). It is equivalent to the fact that the number of lozenge tilings of a hexagon of side-lengths a, b, c, a, b, c (in cyclic order) on the triangular lattice is equal to

$$P(a,b,c) := \frac{\mathrm{H}(a)\,\mathrm{H}(b)\,\mathrm{H}(c)\,\mathrm{H}(a+b+c)}{\mathrm{H}(a+b)\,\mathrm{H}(a+c)\,\mathrm{H}(b+c)}, \tag{1.1}$$

where the hyperfactorial $\mathrm{H}(n)$ is defined by

$$\mathrm{H}(n) := 0!\,1! \cdots (n-1)! \tag{1.2}$$

(see Fig. 1.1 for an example).

The striking elegance of this result compels one to seek similar ones, which could help place it in a broader context. One step in this direction was taken in [5], where it was shown that if instead of the hexagon — which is the region on the triangular lattice traced out by turning 60 degrees at each corner — one considers the figure obtained by turning 120 degrees at each corner — which we called a shamrock in [5] (see Fig. 1.2 for an illustration) — then it is the *exterior* of that region which has, in a certain precise sense, a normalized number of tilings given by a simple product formula analogous to (1.1).

In this paper we introduce a new structure, called a *fern* — an arbitrary string of triangles of alternating orientations that touch at corners and are lined up along a common axis (see Fig. 1.3 for an example) — so that the normalized number of tilings of their exterior is given by a product formula in the style of (1.1). In fact, when the fern has three lobes, one obtains *precisely* formula (1.1)! Therefore, in some sense, this naturally places MacMahon's formula in a sequence of more general exact enumeration formulas. From this point of view, the situation is more satisfying than the one in [5], as a shamrock only has one lobe in addition to a 3-lobe fern.