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Takens' last problem and existence of non-trivial wandering domains

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ABSTRACT

In this paper, we give an answer to a C^r $(2 \leq r < \infty)$ version of the open problem of Takens in [42] which is related to historic behavior of dynamical systems. To obtain the answer, we show the existence of non-trivial wandering domains near a homoclinic tangency, which is conjectured by Colli–Vargas [6, §2]. Concretely speaking, it is proved that any Newhouse open set in the space of C^r-diffeomorphisms on a closed surface is contained in the closure of the set of diffeomorphisms which have non-trivial wandering domains whose forward orbits have historic behavior. Moreover, this result implies an answer in the C^r category to one of the open problems of van Strien [39] which is concerned with wandering domains for Hénon family.

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1. Introduction

1.1. Historic behavior and wandering domains

Consider a dynamical system with a compact state space X, given by a continuous map $\varphi : X \to X$. We say that the forward orbit $\{x, \varphi(x), \varphi^2(x), \dots\}$ of $x \in X$ has *historic behavior* if the partial average

$$\frac{1}{m+1}\sum_{i=0}^{m} \delta_{\varphi^i(x)}$$

does not converge as $m \to \infty$ in the weak topology, where $\delta_{\varphi^i(x)}$ is the Dirac measure on X supported at $\varphi^i(x)$. The terminology of historic behavior was given by Ruelle in [37]. The following is the last open problem presented by Takens (see [42]).

Takens' Last Problem. *Whether are there persistent classes of smooth dynamical systems such that the set of initial states which give rise to orbits with historic behavior has positive Lebesgue measure?*

The first example of historic behavior was given in [19], where it is shown that the logistic family contains elements for which almost all orbits have historic behavior. This was extended to generic full families of unimodal maps, see [8]. While Takens showed in [41] that Bowen's 2-dimensional flow with an attracting heteroclinic loop has a set of positive Lebesgue measure consisting of initial points of orbits with historic behavior, but the property is not preserved under arbitrarily small perturbations of the dynamics. Also, by using Dowker's result [10], Takens showed that the doubling map on the circle persistently has orbits with historic behavior, for which the collection of initial points is a residual subset on the circle, see [42].

In this paper, we obtain an answer to Takens' Last Problem for non-hyperbolic diffeomorphisms having homoclinic tangencies by a different way from the previous works. To solve the problem we use a non-empty connected open set, called a wandering domain, whose images do not intersect each other but are wandering around non-trivial hyperbolic sets. Wandering domains have been studied from the beginning of 20th century. In fact, Bohl [3] in 1916 and Denjoy [9] in 1932 constructed examples of C^1 diffeomorphisms on a circle which have wandering domains whose ω-limit set is a Cantor set. However, it can not be extended to any C^2 as well as C^1 diffeomorphism whose derivative is a function of bounded variation, see in [8]. Subsequently, similar phenomena for high dimensional diffeomorphisms were studied by several authors, for example [25,18,30,4,32,27]. Also, for unimodal as well as multimodal maps on an interval or a circle, the main difficulty in their classification in real analytic category was to show the absence of wandering domains, which were developed by many dynamicists [7,28,2,8,40], see the survey of van Strien [39].