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On the conjecture $\mathcal{O}$ of GGI for G/P

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In this paper, we show that general homogeneous manifolds G/P satisfy Conjecture $\mathcal{O}$ of Galkin, Golyshev and Iritani which 'underlies' Gamma conjectures I and II of them. Our main tools are the quantum Chevalley formula for G/P and a theory on nonnegative matrices including Perron–Frobenius theorem.

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1. Introduction

Let X be a Fano manifold, i.e., a smooth projective variety whose anti-canonical line bundle is ample. The quantum cohomology ring $H^\star(X,\mathbb{C})$[1] of X is a certain deformation of the classical cohomology ring $H^*(X,\mathbb{C})$ (§2.4 below). For $\sigma \in H^\star(X,\mathbb{C})$, define the quantum multiplication operator $[\sigma]$ on $H^\star(X,\mathbb{C})$ by $[\sigma](\tau) = \sigma \star \tau$ for $\tau \in H^\star(X,\mathbb{C})$, where $\star$ denotes the quantum product in $H^\star(X,\mathbb{C})$. Let δ_0 be the absolute value of a maximal modulus eigenvalue of the operator $[c_1(X)]$, where $c_1(X)$ denotes the first Chern class of the tangent bundle of X. In [8], Galkin, Golyshev and Iritani say that X satisfies Conjecture $\mathcal{O}$ if

(1) δ_0 is an eigenvalue of $[c_1(X)]$.
(2) The multiplicity of the eigenvalue δ_0 is one.
(3) If δ is an eigenvalue of $[c_1(TX)]$ such that $|\delta| = \delta_0$, then $\delta = \delta_0\xi$ for some r-th root of unity, where r is the Fano index of X.

In fact, in addition to Conjecture $\mathcal{O}$, Galkin, Golyshev and Iritani proposed two more conjectures called Gamma conjectures I, II, which can be stated under the Conjecture $\mathcal{O}$. Let us briefly introduce Gamma conjectures I, II in order to explain how it underlies them. Consider the quantum connection of Dubrovin

$$\nabla_{z\partial_z} = z\frac{\partial}{\partial z} - \frac{1}{z}(c_1(X)\star) + \mu,$$

acting on $H^*(X,\mathbb{C}) \otimes \mathbb{C}[z,z^{-1}]$, where μ is the grading operator on $H^*(X)$ defined by $\mu(\tau) = (k - \frac{\dim X}{2})\tau$ for $\tau \in H^{2k}(X,\mathbb{C})$. This has a regular singularity at $z=\infty$ and an irregular singularity at $z=0$. Flat sections near $z=\infty$ can be constructed through flat sections near $z=0$ classified by their exponential growth order, and they are put into correspondence with cohomology classes. To be precise, if X satisfies Conjecture $\mathcal{O}$, we can take a flat section $s_0(z)$ with the smallest asymptotics $\sim e^{-\delta_0/z}$ as $z \to +0$ along $\mathbb{R}_{>0}$. We transport $s_0(z)$ to $z=\infty$ and identify the corresponding class A_X called the principal asymptotic class of X. Then Gamma conjecture I states that the cohomology class A_X is equal to the Gamma class $\hat{\Gamma}_X$. Here $\hat{\Gamma}_X := \prod_{i=1}^n \Gamma(1+\vartheta_i) \in H^*(X)$, where ϑ_i are the Chern roots of the tangent bundle TX for $i = 1,...,n$. Under further assumption of semisimplicity of the ring $H^\star(X)$, we can identify cohomology classes A_δ corresponding to each eigenvalue δ in similar way. The classes A_δ form a basis of $H^*(X,\mathbb{C})$. Then Gamma conjecture II, a refinement of a part of Dubrovin's conjecture [3], states that there is an exceptional collection $\{E_\delta \mid \delta \text{ eigenvalues of } [c_1(X)]\}$ of the derived category $D^b_{\mathrm{coh}}(X)$ such that for each δ,

[1] We use this notation for the quantum cohomology ring with the multiplication $\star$, and without quantum variables.