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Quadratic forms and Clifford algebras on derived stacks

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ABSTRACT

In this paper we present an approach to quadratic structures in derived algebraic geometry. We define derived n-shifted quadratic complexes, over derived affine stacks and over general derived stacks, and give several examples of those. We define the associated notion of derived Clifford algebra, in all these contexts, and compare it with its classical version, when they both apply. Finally, we prove three main existence results for derived shifted quadratic forms over derived stacks, define a derived version of the Grothendieck–Witt group of a derived stack, and compare it to the classical one.

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0. Introduction

Prompted by the recent introduction of *symplectic forms* in derived algebraic geometry ([10]), we present a global approach to the theory of *quadratic forms* on very general moduli spaces, called derived stacks. In the case where the derived stack is just the spectrum of a ring k where 2 is invertible, these quadratic forms are defined on complexes C of k-modules, and are maps, in the derived category of k, from $\mathrm{Sym}^2_k C$ to $k[n]$, where $\mathrm{Sym}^2_k(-)$ denotes the derived functor of the second symmetric power over k. There is an obvious notion of non-degeneracy for such a quadratic form, saying that the induced adjoint map $C \to C^\vee[n]$ is an isomorphism in the derived category of k.

The derived features are therefore two: first of all the map is a morphism in the *derived category* of k, and secondly, and most importantly, we allow for a *shift* in the target. These features accommodate for various symmetric shifted duality situations in topology, the motivating one being classical Poincaré duality.

We present a *globalization* of the above particular case to quadratic forms on Modules over a derived stack that uses in an essential way the refined (i.e. homotopical or, equivalently, ∞-categorical) features of derived algebraic geometry. When the Module in question is the tangent complex, we obtain what we call *shifted quadratic stacks*. We remark that the main definitions of derived quadratic forms and derived quadratic stacks are slight modifications of the notion of derived symplectic structure from [10] (without the complication coming from closedness data). In particular, we are able to reproduce in the quadratic case, two of the main existence theorems in [10]: the existence of a shifted quadratic form on the stack of maps from a $\mathcal{O}$-compact, $\mathcal{O}$-oriented derived stack to a shifted quadratic stack (Theorem 4.1), and the existence of a quadratic form on the homotopy fiber product of two null-mappings to a shifted quadratic stack (Theorem 4.7). As a consequence, we get that the derived looping of a shifted quadratic stack decreases the shift by one (Corollary 4.6). We also observe that any shifted symplectic structure gives rise to a shifted quadratic Module.

The third main topic developed in the paper, after the general theory of quadratic forms on derived moduli spaces and the main existence theorems, is the definition and study of a new derived version of the *Clifford algebra* associated to a shifted quadratic Module over a derived stack (so, in particular, to any shifted quadratic stack). We prove various basic properties of this derived Clifford algebra and give a theorem comparing it to the classical Clifford algebra, when they are both defined: the classical Clifford algebra happens to be the truncation at H^0 of the derived one (see Corollary 2.12). We also introduce the notion of a shifted derived version of the Grothendieck–Witt