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Finite range perturbations of finite gap Jacobi and CMV operators

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ABSTRACT

Necessary and sufficient conditions are presented for a measure to be the spectral measure of a finite range perturbation of a Jacobi or CMV operator from a finite gap isospectral torus. The special case of eventually periodic operators solves an open problem of Simon [28, D.2.7].

We also solve the inverse resonance problem: it is shown that an operator is completely determined by the set of its eigenvalues and resonances, and we provide necessary and sufficient conditions on their configuration for such an operator to exist.

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1. Introduction

By a Jacobi operator/matrix we will call a bounded Hermitian operator on $\ell_2(\mathbb{Z}_+)$ of the form

$$\mathcal{J} = \begin{pmatrix} b_1 & a_1 & 0 & \\ a_1 & b_2 & a_2 & \ddots \\ 0 & a_2 & b_3 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}. \tag{1.1}$$

Any operator of the form (1.1) will be denoted by $\mathcal{J}[a_n, b_n]_{n=1}^{\infty}$. Sequences $\{a_n\}$, $\{b_n\}$ are called the Jacobi parameters of $\mathcal{J}$. We always assume these are bounded sequences, and $a_n > 0$, $b_n \in \mathbb{R}$ for all n.

Associated to $\mathcal{J}$, we have μ, the spectral measure of $\mathcal{J}$ with respect to the vector $e_1 := (1, 0, 0, \dots)^T$ (which is cyclic since all $a_j > 0$):

$$\int_{\mathbb{R}} f(x) d\mu(x) = \langle e_1, f(\mathcal{J}) e_1 \rangle. \tag{1.2}$$

Conversely, given any probability measure μ with compact but not finite support in $\mathbb{R}$, we can form the sequence of orthonormal polynomials which satisfy the three-term recurrence relation with the coefficients $\{a_n, b_n\}_{n=1}^{\infty}$ from (1.1).

In this paper we will consider only measures with essential support equal to a finite gap set

$$\mathfrak{e} = \bigcup_{j=1}^{l+1} [\alpha_j, \beta_j], \quad \alpha_1 < \beta_1 < \alpha_2 < \dots < \alpha_{l+1} < \beta_{l+1}. \tag{1.3}$$

We will refer to each $[\alpha_j, \beta_j]$ as a "band", and to each $[\beta_j, \alpha_{j+1}]$ as a "gap".

Associated to $\mathfrak{e}$ is a natural class of operators called the isospectral torus $\mathcal{T}_\mathfrak{e}$ of Jacobi operators. This includes as special cases the free Jacobi operator when $l = 0$, $\mathfrak{e} = [-2, 2]$ and, more generally, all periodic Jacobi operators when harmonic measures of each $[\alpha_j, \beta_j]$ in $\mathfrak{e}$ are rational. If not all of these harmonic measures are rational, then $\mathcal{T}_\mathfrak{e}$ consists of almost-periodic Jacobi operators.

Operators in the isospectral torus are well-studied by now (see the classical works [31, 9,20,21]). We propose to go one step further and consider their finite range perturbations: take $\mathcal{J} \in \mathcal{T}_\mathfrak{e}$ and change finitely many of its Jacobi coefficients.

Similar constructions are also considered for measures on the unit circle. By a CMV operator/matrix we will call a unitary operator on $\ell_2(\mathbb{Z}_+)$ of the form

$$\mathcal{C} = \operatorname{diag}(\Xi_0, \Xi_2, \Xi_4, \dots) \operatorname{diag}(\Xi_{-1}, \Xi_1, \Xi_3, \dots),$$
$$\text{where } \Xi_n = \begin{pmatrix} \bar\alpha_n & \sqrt{1-|\alpha_n|^2} \\ \sqrt{1-|\alpha_n|^2} & -\alpha_n \end{pmatrix}, \text{ and } \Xi_{-1} = (1).$$