Advances in Mathematics 301 (2016) 258–288

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

An invariant for homogeneous spaces of compact quantum groups

Partha Sarathi Chakraborty [a], Arup Kumar Pal [b,*]

[a] The Institute of Mathematical Sciences, CIT Campus, Taramani, Chennai 600113, India
[b] Indian Statistical Institute, 7, SJSS Marg, New Delhi 100016, India

ARTICLE INFO

Article history:
Received 20 May 2014
Received in revised form 17 June 2016
Accepted 18 June 2016
Available online 29 June 2016
Communicated by Dan Voiculescu

MSC:
58B34
58B32
46L87

Keywords:
Noncommutative geometry
Spectral triple
Dimension
Quantum group
Homogeneous space

ABSTRACT

The central notion in Connes' formulation of noncommutative geometry is that of a spectral triple. Given a homogeneous space of a compact quantum group, restricting our attention to all spectral triples that are 'well behaved' with respect to the group action, we construct a certain dimensional invariant. In particular, taking the (quantum) group itself as the homogeneous space, this gives an invariant for a compact quantum group. Computations of this invariant in several cases, including all type A quantum groups, are given.

1. Introduction

Geometry can be broadly interpreted as the study of cycles and their intersection properties in some suitable homology theory. Noncommutative Geometry is no excep-

* Corresponding author.
E-mail addresses: parthac@imsc.res.in (P.S. Chakraborty), arup@isid.ac.in (A.K. Pal).

http://dx.doi.org/10.1016/j.aim.2016.06.013
0001-8708/

tion. In Alain Connes' interpretation, noncommutative geometry is the study of spectral triples or unbounded Kasparov-modules with finer properties [6,8]. Often these finer properties encode information about metric, dimension etc. In fact for an unbounded K-cycle to encode useful information, it is not always necessary that the cycle should be homologically nontrivial. A cycle may be homologically trivial but still it may contain metric or dimensional information. One prime example is the Laplacian on odd dimensional manifolds. Study of cycles not necessarily nontrivial is not new. Apart from Connes (see for example [7]), it also includes Voiculescu's work [24,25] on norm ideal perturbations or Rieffel's work [22] on extending the notion of metric spaces. Voiculescu answers the question of existence of bounded K-cycles in a given representation of a C^*-algebra. He does not comment on the nontriviality of the K-cycle as a K-homology class. Rieffel uses spectral triples to produce compact quantum metric spaces but in his construction nontriviality does not play any major role. Here we take a similar approach. We utilize the notion of spectral triples to produce dimensional invariants for ergodic C^*-dynamical systems. However, we should emphasize a crucial distinction between the above cited works and the present one. In Rieffel's work, Dirac operators are used as a source to produce compact quantum metric spaces; but there are situations [2] where one produces compact quantum metric spaces even without using spectral triples. Whereas in our case, the concept of spectral triple is used in an essential manner, not as a source of examples.

Origin of the present paper lies in the search for nontrivial spectral triples for the quantum $SU(2)$ [3] and quantum spheres [5]. Instead of KK-theoretic machinery, our method tries to characterize equivariant spectral triples. In the process, one observes that even without the condition of nontriviality of the corresponding K-cycle, in certain cases there are canonical spectral triples encoding essential information about the space. That leads to the present invariant.

Let us recall the classical fact that if M is a d-dimensional compact Riemannian manifold and we have an elliptic operator D of order 1, then the n-th singular value of $|D|$ grows like $n^{1/d}$. Using this Connes has indicated how to define the dimension of a spectral triple. This observation can be utilized to obtain invariants for C^*-algebras provided one could associate natural spectral triples with them. But unlike the classical case it is difficult to define natural spectral triples for C^*-algebras. As one tries to answer this question one faces two main difficulties: one, it is difficult to get hold of a canonical representation other than the GNS representation, which normally is too big; two: once a 'natural' representation has been identified, it is impossible to classify all spectral triples so as to be able to extract any meaningful quantity out of them. Another important point pertaining to the last problem above is that when considering a spectral triple, one should look at a spectral triple for some dense *-subalgebra of the C^*-algebra, but in general there is no canonical choice of a dense *-subalgebra; but properties of the behavior of the spectral triple are sensitive to this choice.

If we restrict our attention to homogeneous spaces of compact quantum groups, or which is the same thing, to ergodic C^*-dynamical systems, then these problems can be