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Local positivity in terms of Newton–Okounkov bodies

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A B S T R A C T

In recent years, the study of Newton–Okounkov bodies on normal varieties has become a central subject in the asymptotic theory of linear series, after its introduction by Lazarsfeld–Mustaţă and Kaveh–Khovanskii. One reason for this is that they encode all numerical equivalence information of divisor classes (by work of Jow). At the same time, they can be seen as local positivity invariants, and Küronya–Lozovanu have studied them in depth from this point of view.

We determine what information is encoded by the set of all Newton–Okounkov bodies of a big divisor with respect to flags centered at a fixed point of a surface, by showing that it determines and is determined by the numerical equivalence class of the divisor up to negative components in the Zariski decomposition that do not go through the fixed point.

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1. Introduction

Newton–Okounkov bodies. Inspired by the work of A. Okounkov [12], R. Lazarsfeld and M. Mustaţă [11] and independently K. Kaveh and A. Khovanskii [6] introduced Newton–Okounkov bodies as a tool in the asymptotic theory of linear series on normal varieties, a tool which proved to be very powerful and in recent developments of the theory has gained a central role. An excellent introduction to the subject — not exhaustive due to the rapid development of the theory — can be found in the review [1] by S. Boucksom.

Newton–Okounkov bodies are defined as follows. Let X be a normal projective variety of dimension n. A flag of irreducible subvarieties

$$Y_\bullet = \{X = Y_0 \supset Y_1 \supset \cdots \supset Y_n = \{p\}\}$$

is called *full* and *admissible* if Y_i has codimension i in X and is smooth at the point p. p is called the *center* of the flag. For every non-zero rational function $\phi \in K(X)$, write $\phi_0 = \phi$, and for $i = 1, \ldots, n$

$$\nu_i(\phi) \stackrel{\text{def}}{=} \operatorname{ord}_{Y_i}(\phi_{i-1}) \ , \qquad \phi_i \stackrel{\text{def}}{=} \left. \frac{\phi_{i-1}}{g_i^{\nu_i(\phi)}} \right|_{Y_i} , \tag{$*$}$$

where g_i is a local equation of Y_i in Y_{i-1} around p (this makes sense because the flag is admissible). The sequence $\nu_{Y_\bullet} = (\nu_1, \nu_2, \ldots, \nu_n)$ determines a rank n discrete valuation $K(X)^* \longrightarrow \mathbb{Z}^n_{\text{lex}}$ with center at p [16].

Definition 1. If X is a normal projective variety, D a big Cartier divisor on it, and $Y_\bullet$ an admissible flag, the Newton–Okounkov body of D with respect to $Y_\bullet$ is

$$\Delta_{Y_\bullet}(D) \stackrel{\text{def}}{=} \overline{\left\{ \frac{\nu_{Y_\bullet}(\phi)}{k} \,\middle|\, \phi \in H^0(X, \mathcal{O}_X(kD)),\, k \in \mathbb{N} \right\}} \subset \mathbb{R}^n,$$

where $\overline{\{\cdot\}}$ denotes the closure with respect to the usual topology of $\mathbb{R}^2$. Although not obvious from this definition, $\Delta_{Y_\bullet}(D)$ is convex and compact, with nonempty interior, i.e., a body (see [6,11,1]). A. Küronya, V. Lozovanu and C. Maclean [9] have shown that it is a polygon if X is a surface, and that in higher dimensions it can be non-polyhedral, even if X is a Mori dream space. The definition can be carried over to the more general setting of graded linear series, and also to $\mathbb{Q}$ or $\mathbb{R}$-divisors; in the absence of some bigness condition, $\Delta_{Y_\bullet}(D)$ may fail to have top dimension.

By [11, Proposition 4.1], $\Delta_{Y_\bullet}(D)$ only depends on the numerical equivalence class of D. S.Y. Jow proved in [5] that the set of all Newton–Okounkov bodies works as a complete set of numerical invariants of D, in the sense that, if D' is another big Cartier divisor with $\Delta_{Y_\bullet}(D) = \Delta_{Y_\bullet}(D')$ for all flags $Y_\bullet$, then D and D' are numerically equivalent.

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