Advances in Mathematics 301 (2016) 499–548

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Entropy conditions in two weight inequalities for singular integral operators

Sergei Treil [1], Alexander Volberg [*,2]

ARTICLE INFO

Article history:
Received 11 March 2015
Received in revised form 22 June 2016
Accepted 23 June 2016
Available online 5 July 2016
Communicated by Charles Fefferman

MSC:
42B20
42B35
47A30

Keywords:
Calderón–Zygmund operators
Bellman function
Bump conditions

ABSTRACT

A new type of "bumping" of the Muckenhoupt A_2 condition on weights is introduced. It is based on bumping the entropy integral of the weights. In particular, one gets (assuming mild regularity conditions on the corresponding Young functions) the bump conjecture proved in [20,25] as a corollary of entropy bumping. But our entropy bumps cannot be reduced to the bumping with Orlicz norms in the solution of bump conjecture, they are effectively smaller. Henceforth, we get somewhat stronger results than the one that solves the bump conjecture in [20,25]. New results concerning one sided bumping conjecture are obtained. All the results hold in the general non-homogeneous situation.

1. Introduction

The original question about two weight estimates for singular integral operators is to find a necessary and sufficient condition on the weights (non-negative locally integrable functions) w and v such that a Calderón–Zygmund operator $T : L^p(w) \to L^p(v)$ is

* Corresponding author.
E-mail address: volberg@math.msu.edu (A. Volberg).

[1] ST is partially supported by the NSF grant DMS-1301579.
[2] AV is partially supported by the NSF grant DMS-1265549 and by the Hausdorff Institute for Mathematics, Bonn, Germany.

http://dx.doi.org/10.1016/j.aim.2016.06.021
0001-8708/

bounded, i.e. the inequality

$$\int |Tf|^p v dx \le C \int |f|^p w dx \qquad \forall f \in L^p(w) \tag{1.1}$$

holds.

In the one weight case $w = v$ the famous Muckenhoupt condition is necessary and sufficient for (1.1)

$$\sup_I \left(|I|^{-1} \int_I w dx \right) \left(|I|^{-1} \int_I w^{-p'/p} dx \right)^{p/p'} < \infty \tag{A_p}$$

where the supremum is taken over all cubes I, and $1/p + 1/p' = 1$. More precisely, this condition is sufficient for all Calderón–Zygmund operators, and is also necessary for classical (interesting) Calderón–Zygmund operators, such as Hilbert transform, Riesz transform (vector-valued, when all Riesz transforms are considered together), Beurling–Ahlfors operator.

Inequality (1.1) is equivalent to the boundedness of the operator $M_{v^{1/p}} T M_{w^{-1/p}}$ in the non-weighted L^p; here M_φ is the multiplication operator, $M_\varphi f = \varphi f$. Denoting $u = w^{-p'/p}$ we can rewrite the problem in the symmetric form as the L^p boundedness of $M_{v^{1/p}} T M_{u^{1/p'}}$.

So the problem can be stated as: *Describe all weights (i.e. non-negative functions) u, v such that the operator $M_{v^{1/p}} T M_{u^{1/p'}}$ is bounded in (the non-weighted) L^p.*

The boundedness of $M_{v^{1/p}} T M_{u^{1/p'}}$ means that

$$\int \left| T(u^{1/p'} g) \right|^p v dx \le \int |g|^p dx \qquad \forall g \in L^p,$$

which after denoting $g = u^{1/p} f$ can be rewritten as

$$\int |T(uf)|^p \, v dx \le C \int |f|^p u dx. \tag{1.2}$$

Such symmetric formulation is well known since the 80s, [9], [13], and for the two weight setting it looks more natural. In particular, if T is an integral operator, then the integration in the operator is performed with respect to the same measure udx as in the domain. This simplifies the problem, because it eliminates the third measure (the Lebesgue measure) from the considerations.

Note also, that this formulation is formally more general than (1.1), because in (1.1) it is usually assumed that $w (= u^{-p/p'})$ is locally integrable, while in (1.2) we only assume local integrability of u; in particular, u can be zero on a set of positive measure.

It is usually assumed that in (1.1) v and w are locally integrable, but for (1.2) (respectively (1.1)) to hold for interesting operators (Hilbert Transform, vector Riesz Transform,