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Riesz transforms and fractional integration for orthogonal expansions on spheres, balls and simplices ☆

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This paper studies the Hardy–Littlewood–Sobolev (HLS) inequality and the Riesz transforms for fractional integration associated to weighted orthogonal polynomial expansions on spheres, balls and simplexes with weights being invariant under a general finite reflection group on $\mathbb{R}^d$. The sharp index for the validity of the HLS inequality is determined and the L^p-boundedness of the Riesz transforms is established. In particular, our results extend a classical inequality of Muckenhoupt and Stein on conjugate ultraspherical polynomial expansions. Our idea is based on a new decomposition of the Dunkl–Laplace–Beltrami operator on the sphere and some sharp asymptotic estimates of the weighted Christoffel functions.

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1. Introduction and main results

The classical Hardy–Littlewood–Sobolev (HLS) fractional integration theorem states that if $0 < \alpha < d$ and $1 < p \le q < \infty$, then the HLS inequality,

$$\|(-\Delta)^{-\alpha/2} f\|_{L^q(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}, \quad \forall f \in L^p(\mathbb{R}^d), \tag{1.1}$$

holds if and only if $\alpha = d(\frac{1}{p} - \frac{1}{q})$ (see [38, Chapter V]), where $\partial_j = \frac{\partial}{\partial x_j}$ and $(-\Delta)^\beta$ denotes the fractional power of the Laplacian $\Delta = \sum_{j=1}^d \partial_j^2$. This theorem implies the Sobolev embedding theorem essentially by the relationship between the Riesz transforms $R_j = \partial_j(-\Delta)^{-\frac{1}{2}}$, $j = 1, 2\cdots, d$ and the fractional integral operators $(-\Delta)^{-\alpha/2}$ (i.e. the Riesz potentials). The HLS inequality and the Riesz transforms on $\mathbb{R}^d$ have been extended to many different settings with fractional integration being mostly defined via orthogonal expansions or distributional Fourier transform (see, for instance, [1–3,6,9,27,32,37,38,36,40]).

In this paper, we will study the HLS inequality and the Riesz transforms for fractional integration associated to weighted orthogonal polynomial expansions (WOPEs) on the sphere $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : \|x\| = 1\}$, on the ball $\mathbb{B}^d := \{x \in \mathbb{R}^d : \|x\| \le 1\}$ and on the simplex $\mathbb{T}^d := \{x \in \mathbb{R}^d : x_1, \cdots, x_d \ge 0, |x| \le 1\}$ with weights being invariant under a general finite reflection group on $\mathbb{R}^d$. Here and throughout the paper, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$, and $|x| := \sum_{j=1}^d |x_j|$ denotes the ℓ^1-norm of $\mathbb{R}^d$. In this introduction we shall describe our main results for WOPEs on the sphere $\mathbb{S}^{d-1}$ with a "minimum" of definitions. Necessary details and appropriate definitions will be given in the next section.

Let $G \subset O(d)$ be a finite reflection group on $\mathbb{R}^d$. For $v \in \mathbb{R}^d \setminus \{0\}$, we denote by σ_v the reflection with respect to the hyperplane perpendicular to v; that is,

$$\sigma_v x = x - \frac{2\langle x, v\rangle}{\|v\|^2} v, \quad x \in \mathbb{R}^d,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^d$. Let $\mathcal{R}$ be the root system of G, normalized so that $\langle v, v\rangle = 2$ for all $v \in \mathcal{R}$, and fix a positive subsystem $\mathcal{R}_+$ of $\mathcal{R}$. It is known that (see, for instance, [33]) the set of reflections in G coincides with the set $\{\sigma_v : v \in \mathcal{R}_+\}$, which also generates the group G. The dimension of the linear subspace of $\mathbb{R}^d$ spanned by all elements from the root system $\mathcal{R}$ is called the rank of $\mathcal{R}$ and is denoted by rank$(\mathcal{R})$. Let $\kappa : \mathcal{R} \to [0, \infty)$, $v \mapsto \kappa_v = \kappa(v)$ be a nonnegative multiplicity function on $\mathcal{R}$ (i.e., a nonnegative G-invariant function on $\mathcal{R}$). Let h_κ denote the weight function on $\mathbb{R}^d$ defined by

$$h_\kappa(x) := \prod_{v\in\mathcal{R}_+} |\langle x, v\rangle|^{\kappa_v}, \quad x \in \mathbb{R}^d. \tag{1.2}$$

The function h_κ is G-invariant and homogeneous of degree $|\kappa| := \sum_{v\in\mathcal{R}_+} \kappa_v$.