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The spherical convex floating body

Florian Besau [a,1], Elisabeth M. Werner [b,*,2]

[a] *Institute of Discrete Mathematics and Geometry at Vienna University of Technology, Vienna 1040, Austria*

[b] *Department of Mathematics at Case Western Reserve University, Cleveland, OH 44106, United States*

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ABSTRACT

For a convex body on the Euclidean unit sphere the spherical convex floating body is introduced. The asymptotic behavior of the volume difference of a spherical convex body and its spherical floating body is investigated. This gives rise to a new spherical area measure, the floating area. Remarkably, this floating area turns out to be a spherical analogue to the classical affine surface area from affine differential geometry. Several properties of the floating area are established.

1. Introduction

The floating body appeared first in the work of C. Dupin [16] in 1822. By the end of the 20th century this basic notion witnessed a surge in interest. In 1990, a seminal new definition was given by C. Schütt and E. M. Werner [72] and independently by I. Bárány

* Corresponding author.
E-mail addresses: florian.besau@tuwien.ac.at (F. Besau), elisabeth.werner@case.edu (E.M. Werner).

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and D. G. Larman [8]. They introduced the *convex* floating body as the intersection of all halfspaces whose hyperplanes cut off a set of fixed volume of a convex body (compact convex set). In contrast to the original definition, the convex floating body is always convex and coincides with Dupin's floating body if it exists.

One of the many reasons that the (convex) floating body has attracted considerable interest in recent decades is that it allows extensions of the classical notion of affine surface area to general convex bodies in all dimensions. Indeed, as was shown by K. Leichtweiß [38] and C. Schütt and E. M. Werner [72], the affine surface area arises as a limit of the volume difference of the convex body and its floating body.

Affine surface area was introduced by W. Blaschke [11] in 1923 for smooth convex bodies in Euclidean space of dimensions 2 and 3. Even though it proved much more difficult to extend affine surface area to general convex bodies than other notions, like surface area measures or curvature measures, successively such extensions were achieved. Aside from the aforementioned successful approach via the (convex) floating body, E. Lutwak [46] was able to provide an extension in 1991 by a completely different method and also proved the long conjectured upper semicontinuity of affine surface area.

As the name suggests, affine surface area is invariant under volume preserving affine transformations. Furthermore it is a valuation on the space of convex bodies and, as mentioned, upper semicontinuous. M. Ludwig and M. Reitzner [43] proved that these three properties essentially characterize affine surface area. They showed that a valuation on convex bodies that is upper semicontinuous and invariant under volume preserving affine transformations is a linear combination of affine surface area, volume, and the Euler characteristic. Building on results of M. Ludwig and M. Reitzner [44], this result was recently considerably strengthened by C. Haberl and L. Parapatits [30].

Affine surface area is among the most powerful tools in equiaffine differential geometry (see B. Andrews [5,6], A. Stancu [76,77], M. Ivaki [33] and M. Ivaki and A. Stancu [34]). It appears naturally as the Riemannian volume of a smooth convex hypersurface with respect to the affine metric (or Berwald–Blaschke metric), see e.g. the thorough monograph of K. Leichtweiß [39] or the book by K. Nomizu and T. Sasaki [56]. In particular the upper semicontinuity proved to be critical in the solution of the affine Plateau problem by N. S. Trudinger and X.-J. Wang [78].

A variant of the convex floating body provided a geometric interpretation of L_p-affine surface area, see C. Schütt and E. Werner [74]. L_p-affine surface area is a generalization of affine surface area in the L_p-Brunn–Minkowski theory introduced by E. Lutwak [47] (also see D. Hug [32] and M. Meyer and E. M. Werner [55]). M. Ludwig and M. Reitzner [44] recently generalized L_p-affine surface area to Orlicz affine surface area.

One of the fundamental inequalities for affine surface area is the affine isoperimetric inequality (see W. Blaschke [11], L. A. Santaló [61] and C. M. Petty [59]) which states that, among all convex bodies with fixed volume, ellipsoids have the largest affine surface area. This inequality is related to various other inequalities, see E. Lutwak [45] and E. Lutwak, D. Yang and G. Zhang [50]. In particular, the affine isoperimetric inequality implies the Blaschke–Santaló inequality and it proved to be the key ingredient in the