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Dual fans and mirror symmetry

Patrick Clarke

Department of Mathematics, Drexel University, Philadelphia, PA 19104, United States

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ABSTRACT

We show that the mirror constructions of Greene–Plesser, Berglund–Hübsch, Batyrev, Batyrev–Borisov, Givental and Hori–Vafa can be expressed in terms of what we call dual fans. To do this, we associate to a pair of dual fans a pair of toric Landau–Ginzburg models, and we describe a process by which each of the mirror constructions listed also produces a pair of toric Landau–Ginzburg models. Replacing mirror pairs by toric Landau–Ginzburg models is reversible, and our main result is that the dual fan models and the mirror pairs models coincide.

1. Introduction

In this paper we unify the mirror constructions of Greene–Plesser [17], Berglund–Hübsch [5], Batyrev [1], Batyrev–Borisov [3],[1] Givental [16], and Hori–Vafa [18] into a single notion of duality based on what we call **dual fans**. Fan duality is the fullest form of the phenomenon observed by Clarke [8], and this point of view has already proven useful. For example, it has led to considerable simplifications by Clarke [9] of the birationality results of Shoemaker [22] and Kelly [19].

E-mail address: pclarke@math.drexel.edu.

[1] As a consequence, Batyrev [2] and Borisov [6] are also described this way.

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The idea of dual fans is simply that of a pair of fans which pair positively (Definition 1.1). This turns out to be all that is needed to produce a pair of toric Landau–Ginzburg models for which the geometry of one is captured by the superpotential of the other (Definition 1.2). For us a **toric Landau–Ginzburg model** is a **toric variety** X equipped with a morphism

$$W\colon X \to \mathbf{A}^1$$

called the **superpotential**. The simplest example of this is Batyrev's construction of the mirror to a complete toric variety (Example 1.3).

The different toric mirror constructions listed above present the mirror pairs in several different ways. In order to put these on a common footing and present them in a way that facilitates comparison to the dual fan Landau–Ginzburg models, we derive from each mirror construction a pair of **auxiliary toric Landau–Ginzburg** models (Definition 1.4). Although these Landau–Ginzburg models are derived from the original mirror pairs, each pair is equivalent to its original in the sense that the original pairs can be recovered (Remark 1.5). In physics, these auxiliary toric LGs arise in the context of the Calabi–Yau/Landau–Ginzburg correspondence of Vafa–Warner [23] (see also Witten [24]).

Our main theorem (Theorem 1.6) makes explicit the connection between the Landau–Ginzburg models constructed from dual fans and those of the mirror pairs. In brief, the fans describing the geometry of mirror auxiliary toric Landau–Ginzburg models are dual, and the auxiliary toric Landau–Ginzburg models are exactly the pair of toric Landau–Ginzburg models defined from the extracted fans.

1.1. Precise constructions and results

Here we state precisely the main constructions and results of the paper. A review of the relevant toric geometry and the proofs of the results are found in the main body of the paper.

Dual fans. Dual fans are a pair of fans in dual spaces which pair positively. This is the basic data needed to form a pair of toric Landau–Ginzburg models in which the geometry of one is reflected in the superpotential of the other. We define these objects and describe the resulting Landau–Ginzburg models here.

Definition 1.1 *(Dual fans).* Dual fans Σ in N, Σ' in M are fans of strongly convex rational polyhedral cones such that

- M and N are free abelian groups of finite rank and dual to each other, and
- $0 \leq \langle m, n\rangle$ for any $m \in |\Sigma'|$ and $n \in |\Sigma|$.

The first step in making contact between the mirror constructions above and pairs of dual fans is taken through the formation a toric **Landau–Ginzburg model** and its **dual**.