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Isosystolic inequalities for optical hypersurfaces[☆]

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ABSTRACT

We explore a natural generalization of systolic geometry to Finsler metrics and optical hypersurfaces with special emphasis on its relation to the Mahler conjecture and the geometry of numbers. In particular, we show that if an optical hypersurface of contact type in the cotangent bundle of the 2-dimensional torus encloses a volume V, then it carries a periodic characteristic whose action is at most $\sqrt{V/3}$. This result is deduced from an interesting dual version of Minkowski's lattice-point theorem: if the origin is the unique integer point in the interior of a planar convex body, the area of its dual body is at least 3/2.

> *Never consider a convex body without considering its dual at the same time.*
>
> [I.M. Gelfand]

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1. Introduction

Minkowski's first theorem in the geometry of numbers states that *if the volume of a* 0*-symmetric convex body in* $\mathbb{R}^n$ *is at least* 2^n*, the body contains a non-zero integer point.* On the other hand, it is easy to find asymmetric convex bodies of arbitrary large volume that contain the origin and no other integer point. It is tempting to say something about the geometry of such bodies. For example, it is known that they must be flat in some lattice direction (see [26] and [9]). In this paper we show that the volume of their duals cannot be arbitrarily small. In fact, the interplay between contact and systolic geometry studied in [2] suggests the following sharp inequality:

Conjecture I. *If the interior of a convex body in* $\mathbb{R}^n$ *contains no integer point other than the origin, then the volume of its dual body is at least* $(n+1)/n!$*. Moreover, equality holds if and only if the convex body is a simplex such that the integer points on its boundary are precisely its vertices.*

Another formulation of the conjecture that seems more elementary is as follows: *if every integer hyperplane* $m_1x_1 + \cdots + m_nx_n = 1$*—where the* m_i *are integers not all equal to zero—intersects a convex body* $K \subset \mathbb{R}^n$*, then the volume of* K *is at least* $(n+1)/n!$*.*

We prove both the two-dimensional case of the conjecture and its asymptotic version:

Theorem I. *The area of a convex body in the plane that intersects every integer line* $mx + ny = 1$ *is at least* $3/2$*. Moreover, equality holds only for the triangle with vertices* $(1,0)$*,* $(0,1)$*,* $(-1,-1)$ *and its images under* $GL(2,\mathbb{Z})$ *(see Fig. 1).*

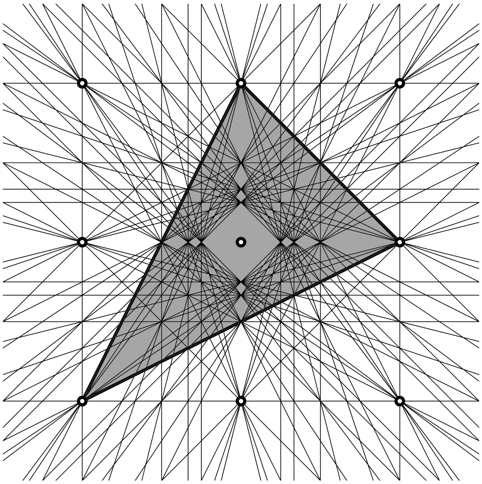

Fig. 1. Integer lines $mx + ny = 1$ with $-4 \leq m, n \leq 4$ and a convex body of minimal area that intersects every integer line.