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A combinatorial geometric Satake equivalence

Joel Kamnitzer

Department of Mathematics, University of Toronto, Toronto, ON, Canada

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ABSTRACT

The geometric Satake correspondence provides an equivalence of categories between the Satake category of spherical perverse sheaves on the affine Grassmannian and the category of representations of the dual group. In this note, we define a combinatorial version of the Satake category using irreducible components of fibres of the convolution morphism. We then prove an equivalence of coboundary categories between this combinatorial Satake category and the category of crystals of the dual group.

1. Introduction

Let G be a complex reductive group and let $G^\vee$ be its Langlands dual group.

The geometric Satake equivalence of Lusztig [10], Ginzburg [5], and Mirkovic–Vilonen [11] provides a description of the representation theory of $G^\vee$ in terms of the topology of the affine Grassmannian $\mathrm{Gr} = G((t))/G[[t]]$ of G. More precisely, the above authors defined a symmetric monoidal category of $G[[t]]$-equivariant perverse sheaves on Gr (known as the Satake category) and then proved that the Satake category is equivalent to the category of representations of $G^\vee$.

E-mail address: jkamnitz@math.toronto.edu.

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In this paper, we define a combinatorial version of the Satake category and then prove that it is equivalent to the category of $G^\vee$-crystals.

1.1. The combinatorial Satake category

To explain our combinatorial Satake category, let us recall that the usual Satake category is a semisimple category whose simple objects are the IC sheaves $\mathrm{IC}(\mathrm{Gr}^\lambda)$ of spherical Schubert varieties. The monoidal structure is defined by convolution and a standard computation shows that

$$\mathrm{IC}(\mathrm{Gr}^\lambda) \otimes \mathrm{IC}(\mathrm{Gr}^\mu) \cong \bigoplus_\nu \mathrm{IC}(\mathrm{Gr}^\nu) \otimes H_{2\langle \lambda+\mu-\nu, \rho\rangle}(m^{-1}(t^\nu)),$$

where $m : \mathrm{Gr}^\lambda \tilde{\times} \mathrm{Gr}^\mu \to \mathrm{Gr}$ is the convolution morphism. The vector space $H_{2\langle \lambda+\mu-\nu, \rho\rangle}(m^{-1}(t^\nu))$ has a natural basis consisting of the set $C^\nu_{\lambda\mu}$ of top-dimensional irreducible components of $m^{-1}(t^\nu)$.

Thus, we combinatorialize the Satake category by defining a semisimple monoidal category $\mathcal{CS}$ where the tensor product is defined using the sets $C^\nu_{\lambda\mu}$. We then equip this category with associativity and commutativity constraints. For associativity, we use iterated convolutions, and for commutativity, we use a certain automorphism of G, inspired by an idea of Beilinson–Drinfeld [2].

1.2. The equivalence with crystals

One might imagine that $\mathcal{CS}$ is equivalent to the Satake category and thus to $Rep\, G^\vee$ (at least as a monoidal category), since it is defined using the same data. In this paper, we show that this is not so — rather it is equivalent to the category of $G^\vee$-crystals, which is a combinatorial version of the representation category of $G^\vee$. We are able to prove an equivalence between these two categories using the work of Braverman–Gaitsgory [4].

Theorem 1.1. *There is an equivalence of coboundary categories $\mathcal{CS} \cong G^\vee$-$\mathcal{C}rys$.*

It is not immediately obvious that $\mathcal{CS}$ is a coboundary category, but this follows from the theorem. The coboundary category structure of $\mathcal{CS}$ will be further explored in [6].

1.3. Ramifications

Though we think of $G^\vee$-$\mathcal{C}rys$ as a combinatorial version of $Rep\, G^\vee$, it is genuinely different. More precisely, suppose we form $G^\vee$-$\mathcal{C}rys \otimes \mathbb{C}$; the category where the objects are the same as $G^\vee$-$\mathcal{C}rys$ and where the morphism sets have been $\mathbb{C}$-linearly extended. Then $G^\vee$-$\mathcal{C}rys \otimes \mathbb{C}$ is certainly equivalent to $Rep\, G^\vee$ as a category, but it is not equivalent as a monoidal category, not even for $G^\vee = SL_2$, as can be seen by considering the