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Tropicalization of del Pezzo surfaces

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A B S T R A C T

We determine the tropicalizations of very affine surfaces over a valued field that are obtained from del Pezzo surfaces of degree 5, 4 and 3 by removing their (−1)-curves. On these tropical surfaces, the boundary divisors are represented by trees at infinity. These trees are glued together according to the Petersen, Clebsch and Schläfli graphs, respectively. There are 27 trees on each tropical cubic surface, attached to a bounded complex with up to 73 polygons. The maximal cones in the 4-dimensional moduli fan reveal two generic types of such surfaces.

1. Introduction

A smooth cubic surface X in projective 3-space $\mathbb{P}^3$ contains 27 lines. These lines are characterized intrinsically as the (-1)-curves on X, that is, rational curves of self-intersection -1. The tropicalization of an embedded surface X is obtained directly from the cubic polynomial that defines it in $\mathbb{P}^3$. The resulting tropical surfaces are dual to

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regular subdivisions of the size 3 tetrahedron. These come in many combinatorial types [15, §4.5]. If the subdivision is a unimodular triangulation then the tropical surface is called smooth (cf. [15, Prop. 4.5.1]).

Alternatively, by removing the 27 lines from the cubic surface X, we obtain a very affine surface X^0. In this paper, we study the tropicalization of X^0, denoted $\mathrm{trop}(X^0)$, via the embedding in its intrinsic torus [11]. This is an invariant of the surface X. The (-1)-curves on X now become visible as 27 *boundary trees* on $\mathrm{trop}(X^0)$. This distinguishes our approach from Vigeland's work [27] on the 27 lines on tropical cubics in $\mathbb{TP}^3$. It also highlights an important feature of tropical geometry [17]: there are different tropical models of a single classical variety, and the choice of model depends on what structure one wants revealed.

Throughout this paper we work over a field K of characteristic zero that has a non-archimedean valuation. Examples include the Puiseux series $K = \mathbb{C}\{\{t\}\}$ and the p-adic numbers $K = \mathbb{Q}_p$. We use the term *cubic surface* to mean a marked smooth del Pezzo surface X of degree 3. A *tropical cubic surface* is the intrinsic tropicalization $\mathrm{trop}(X^0)$ described above. Likewise, *tropical del Pezzo surface* refers to the tropicalization $\mathrm{trop}(X^0)$ for degree ≥ 4. Here, the adjective "tropical" is used solely for brevity, instead of the more accurate "tropicalized" used in [15]. We do not consider non-realizable tropical del Pezzo surfaces, nor tropicalizations of surfaces defined over a field K with positive characteristic.

The moduli space of cubic surfaces is four-dimensional, and its tropical version is the four-dimensional *Naruki fan*. This was constructed combinatorially by Hacking, Keel and Tevelev [11], and it was realized in [20, §6] as the tropicalization of a very affine variety $\mathcal{Y}^0$, obtained from the Yoshida variety $\mathcal{Y}$ in $\mathbb{P}^{39}$ by intersecting with $(K^*)^{39}$. The Weyl group $W(\mathrm{E}_6)$ acts on $\mathcal{Y}$ by permuting the 40 coordinates. The maximal cones in $\mathrm{trop}(\mathcal{Y}^0)$ come in two $W(\mathrm{E}_6)$-orbits. We here compute the corresponding cubic surfaces:

Theorem 1.1. *There are two generic types of tropical cubic surfaces. They are contractible and characterized at infinity by* 27 *metric trees, each having* 10 *leaves. The first type has* 73 *bounded cells,* 150 *edges,* 78 *vertices,* 135 *cones,* 189 *flaps,* 216 *rays, and all* 27 *trees are trivalent. The second type has* 72 *bounded cells,* 148 *edges,* 77 *vertices,* 135 *cones,* 186 *flaps,* 213 *rays, and three of the* 27 *trees have a* 4*-valent node. (For more data see Table 1.)*

Here, by *cones* and *flaps* we mean unbounded 2-dimensional polyhedra that are affinely isomorphic to $\mathbb{R}^2_{\geq 0}$ and $[0,1] \times \mathbb{R}_{\geq 0}$ respectively. The *characterization at infinity* is analogous to that for tropical planes in [12]. Indeed, by [12, Theorem 4.4], every tropical plane L in $\mathbb{TP}^{n-1}$ is given by an arrangement of n boundary trees, each having $n-1$ leaves, and L is uniquely determined by this arrangement. Viewed intrinsically, L is the tropicalization of a very affine surface, namely the complement of n lines in $\mathbb{P}^2$. Theorem 1.1 offers the analogous characterization for the tropicalization of the complement of the 27 lines on a cubic surface.