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Algebra of the infrared and secondary polytopes

M. Kapranov[a,*], M. Kontsevich[b], Y. Soibelman[c]

[a] *Kavli Institute for Physics and Mathematics of the Universe (WPI), 5-1-5 Kashiwanoha, Kashiwa-shi, Chiba, 277-8583, Japan*
[b] *Institut des Hautes Études Scientifiques, 35 Route de Chartres, 91440 Bures-sur-Yvette, France*
[c] *Department of Mathematics, Kansas State University, Cardwell Hall, Manhattan, KS 66506, USA*

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To the memory of Andrei Zelevinsky

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ABSTRACT

We study algebraic structures (L_∞ and A_∞-algebras) introduced by Gaiotto, Moore and Witten in their recent work devoted to certain supersymmetric 2-dimensional massive field theories.

We show that such structures can be systematically produced in any number of dimensions by using the geometry of secondary polytopes, esp. their factorization properties. In particular, in 2 dimensions, we produce, out of a polyhedral "coefficient system", a dg-category R with a semi-orthogonal decomposition and an L_∞-algebra $\mathfrak{g}$. We show that $\mathfrak{g}$ is quasi-isomorphic to the ordered Hochschild complex of R, governing deformations preserving the semi-orthogonal decomposition. This allows us to give a more precise mathematical formulation of the (conjectural) alternative description of the Fukaya–Seidel category of a Kahler manifold endowed with a holomorphic Morse function.

* Corresponding author.
E-mail addresses: mikhail.kapranov@ipmu.jp (M. Kapranov), maxim@ihes.fr (M. Kontsevich), soibel@math.ksu.edu (Y. Soibelman).

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Contents

0. Introduction

The words "algebra of the infrared" in the title refer to the physics paper [10] by Gaiotto, Moore and Witten, to which (or, rather, to a part of which) our article is a mathematical commentary.

In [10] the authors developed an algebraic formalism for the study of certain 2-dimensional massive quantum field theories with $(2, 2)$ supersymmetry. One of the main algebraic structures introduced in the [10] is the L_∞-*algebra of webs*, associated with a generic finite subset $A \subset \mathbb{R}^2$ in the plane. Physically, elements of A correspond to vacua of the theory. A web is a plane graph with faces marked by elements of A with an additional condition on the direction of edges, see §13 below for a review. Further, a choice of a half-plane containing A determines an A_∞-algebra (or an A_∞-category, if one introduces a coefficient system). This A_∞-category has an "upper-triangular structure", i.e. a semi-orthogonal decomposition.

Using certain "moduli spaces of ζ-instantons" the authors of [10] describe a class of deformations of the above A_∞-category which describe the D-brane categories for a particular class of $(2, 2)$ supersymmetric theories: Landau–Ginzburg models. Mathematically, the D-brane A_∞-categories corresponding to LG models are known as Fukaya–Seidel categories [16,23].

We reinterpret and develop the algebraic structures proposed in [10] in a way that allows a generalization to higher dimensions ($\mathbb{R}^d$, $d \geq 2$ instead of just $\mathbb{R}^2$). It turns out that using the dual language of polygons rather than webs, one quickly uncovers certain structures well-known in toric geometry, most notably, *secondary polytopes*, see [11]. One of the subtle and surprising points of the GMW construction is the fact that the differential they define, satisfies $d^2 = 0$. In our approach this fact becomes obvious: the