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Tensor diagrams and cluster algebras ☆

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ABSTRACT

The rings of SL(V) invariants of configurations of vectors and linear forms in a finite-dimensional complex vector space V were explicitly described by Hermann Weyl in the 1930s. We show that when V is 3-dimensional, each of these rings carries a natural cluster algebra structure (typically, many of them) whose cluster variables include Weyl's generators. We describe and explore these cluster structures using the combinatorial machinery of tensor diagrams. A key role is played by the web bases introduced by G. Kuperberg.

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Introduction

Homogeneous coordinate rings of Grassmannians are among the most important examples of cluster algebras. Cluster structures in these rings [23,50] play a prominent role in applications of cluster theory arising in connection with integrable systems, algebraic Lie theory, Poisson geometry, Teichmüller theory, total positivity, and beyond; see, e.g., [21,24,25,29,32] and references therein. Within cluster algebra theory proper, Grassmannians provide the most concrete and accessible examples of naturally defined cluster algebras of infinite mutation type.

Despite their importance, cluster structures on Grassmannians are not well understood at all, apart from a few special cases. Just a tiny subset of their cluster variables have been explicitly described; we do not know which quivers appear in their seeds; we do not understand the structure of their underlying cluster complexes; and so on.

Let $\mathrm{Gr}_{k,N}$ denote the Grassmann manifold of k-subspaces in an N-dimensional complex vector space. The corresponding cluster algebra has finite type (i.e., has finitely many seeds) if and only if $(k-2)(N-k-2) \le 3$. All of the problems mentioned above are open for any Grassmannian of infinite cluster type, so in particular for $k = 3$, $N \ge 9$. (The case $k = 2$ has been well understood since the early days of cluster algebras, see [16, Section 12.2].)

We advocate the point of view that many aspects of cluster structures on Grassmannians are best understood within a broader range of examples coming from classical invariant theory. Recall that the homogeneous coordinate ring of $\mathrm{Gr}_{k,N}$ (with respect to a Plücker embedding) is isomorphic to the ring of $\mathrm{SL}(V)$ invariants of N-tuples of vectors in a k-dimensional complex vector space V. More general rings of $\mathrm{SL}(V)$ invariants of collections of vectors and linear forms have been thoroughly studied by classical invariant