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A notion of the weighted σ_k-curvature for manifolds with density ☆

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ABSTRACT

We propose a natural definition of the weighted σ_k-curvature for a manifold with density; i.e. a triple $(M^n, g, e^{-\phi}\,\mathrm{dvol})$. This definition is intended to capture the key properties of the σ_k-curvatures in conformal geometry with the role of pointwise conformal changes of the metric replaced by pointwise changes of the measure. We justify our definition through three main results. First, we show that shrinking gradient Ricci solitons are local extrema of the total weighted σ_k-curvature functionals when the weighted σ_k-curvature is variational. Second, we characterize the shrinking Gaussians as measures on Euclidean space in terms of the total weighted σ_k-curvature functionals. Third, we characterize when the weighted σ_k-curvature is variational. These results are all analogues of their conformal counterparts, and in the case $k = 1$ recover some of the well-known properties of Perelman's $\mathcal{W}$-functional.

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1. Introduction

First introduced by Viaclovsky [41], the σ_k-curvature is an important Riemannian invariant studied in conformal geometry which has many important connections to questions in analysis, topology and geometry. For example, the study of the σ_k-curvature has led to the development of techniques for solving and characterizing solutions of fully-nonlinear second order PDE [4,15,32,40], it has led to the development of new geometric functional inequalities [25,40], it is closely related to the Euler characteristic in even dimensions [41], a fact which has been particularly fruitful in the study of four-manifolds [16,26], and it can be used to characterize spaceforms [27] and give criteria for the existence of metrics with positive Ricci curvature [15,24,27]. In the special case $k = 1$, the σ_k-curvature is a nonzero constant multiple of the scalar curvature, an object whose analytic, topological and geometric significance is much better understood; cf. [23,31,39].

First introduced by Hamilton [28], the Ricci flow is another important tool in Riemannian geometry with many analytic, topological, and geometric applications. Perhaps the most famous is Perelman's resolution [36] of the Poincaré Conjecture, the proof of which relies heavily on all of these components. In particular, we highlight the role of Perelman's $\mathcal{W}$-functional

$$\mathcal{W}(g,\phi,\tau) := \int_M \left[\tau(R+|\nabla\phi|^2)+\phi-n\right](4\pi\tau)^{-\frac{n}{2}}e^{-\phi}\,\mathrm{dvol}\,.$$

Perelman showed that the $\mathcal{W}$-functional and the ν-entropy $\nu(g)$ — defined by minimizing the $\mathcal{W}$-functional over all pairs (ϕ,τ) such that $\int(4\pi\tau)^{-\frac{n}{2}}e^{-\phi}\,\mathrm{dvol} = 1$ — are nondecreasing along the Ricci flow, and moreover, if $\nu(g)$ is finite, then (M^n,g) is κ-noncollapsed [36]. In this way the $\mathcal{W}$-functional and the ν-entropy are analogous to the total scalar curvature functional and the Yamabe constant, respectively. Indeed, Perelman described (cf. Section 3) multiple ways in which the $\mathcal{W}$-functional should be regarded as the appropriate notion of the total scalar curvature functional on a manifold with density, while positivity of the Yamabe constant implies noncollapsing via uniform control on the L^2-Sobolev constant. Here, a manifold with density is a triple $(M^n,g,e^{-\phi}\,\mathrm{dvol})$ of a Riemannian manifold (M^n,g) together with a smooth measure $e^{-\phi}\,\mathrm{dvol}$ determined by a function $\phi\in C^\infty(M)$ and the Riemannian volume element dvol determined by g. The relationship between the $\mathcal{W}$-functional and the Yamabe functional can be made more precise through the language of smooth metric measure spaces, wherein one attaches a notion of "dimension" to the measure $e^{-\phi}\,\mathrm{dvol}$ which interpolates between the Riemannian case (when this is the dimension of M) and the case where Perelman's $\mathcal{W}$-functional arises (when this is infinite); one approach to this is presented in [10].

Inspired by the successes of the σ_k-curvatures and the $\mathcal{W}$-functional in Riemannian geometry, we both offer definitions of the weighted σ_k-curvatures on a manifold with