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Resolutions in factorization categories

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A B S T R A C T

Building upon ideas of Eisenbud, Buchweitz, Positselski, and others, we introduce the notion of a factorization category. We then develop some essential tools for working with factorization categories, including constructions of resolutions of factorizations from resolutions of their components and derived functors. Using these resolutions, we lift fully-faithfulness and equivalence statements from derived categories of Abelian categories to derived categories of factorizations. Some immediate geometric consequences include a realization of the derived category of a projective hypersurface as matrix factorizations over a noncommutative algebra and recover of a theorem of Baranovsky and Pecharich.

1. Introduction

Since their introduction by D. Eisenbud [14], matrix factorizations have spread from commutative algebra into a wide range of fields. In theoretical physics, M. Kontsevich

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realized that matrix factorizations represent boundary conditions in Landau–Ginzburg models. In topology, matrix factorizations have been used to create knot and link invariants [17,18]. In algebraic geometry, deep statements tying the geometry of projective hypersurfaces to matrix factorizations of their defining polynomial have been proven by D. Orlov [21]. In addition, through mirror symmetry, matrix factorizations allow access to the structure of Fukaya categories of symplectic manifolds, [27,11,1,28].

The original concept of matrix factorizations can be generalized in various ways, e.g. to the stable module category [10], the category of singularities [20], or, in another direction towards more general spaces [24,13,22].

Much of the task of this paper is to repackage Positselski's ideas towards a general theory of matrix factorizations for any Abelian category, in particular to derive functors of factorizations as one would functors of Abelian categories. To this end, we introduce the notion of a factorization category for a triple $(\mathcal{A}, \Phi, w)$ where $\mathcal{A}$ is an Abelian category, $\Phi : \mathcal{A} \to \mathcal{A}$ is an autoequivalence, and $w : \mathrm{Id} \to \Phi$ is a natural transformation. By appropriately altering $\mathcal{A}$ and setting $w = 0$, one fully recovers the usual construction of the derived category $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$.

As factorization categories can rightly be viewed as a deformation of Φ-twisted, two-periodic chain complexes over $\mathcal{A}$, one should be able to build resolutions in a straightforward manner from resolutions of the components of a factorization. A key development of this paper is to provide a construction of such resolutions, see Theorems 3.8 and 3.11.

Now consider two triples, as above, $(\mathcal{A}, \Phi, w)$ and $(\mathcal{B}, \Psi, v)$, and an additive functor, $\theta : \mathcal{A} \to \mathcal{B}$, such that

$$\theta \circ \Phi \cong \Psi \circ \theta$$

and

$$\theta(w_A) = v_{\theta(A)} : \theta(A) \to \theta(\Phi(A)) \cong \Psi(\theta(A))$$

for all objects, $A \in \mathcal{A}$. Furthermore, assume that θ is left-exact, that $\mathcal{A}$ has small coproducts and enough injectives, and that coproducts of injectives are injective. We can then use these resolutions to prove that if the right derived functor of θ is fully-faithful then so is a "right derived functor" associated to θ between the derived categories of factorizations. Moreover, if the right derived functor of θ is an equivalence induced by Abelian natural transformations, then so is a "right derived functor" associated to θ between the derived categories of factorizations. We can also use these resolutions to construct a spectral sequence computing the morphism spaces in the derived categories of factorizations whose E_1-page consists of Ext-groups between the components of this factorization in the underlying Abelian category.

From these results, we are able to lay much of the groundwork for handling these categories with the bulk of the machinery developed for derived categories. Moreover, one can deduce many results about factorization categories from results about the usual