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Global solutions for a supercritical drift–diffusion equation

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A B S T R A C T

We study the global existence of solutions to a one-dimensional drift–diffusion equation with logistic term, generalizing the classical parabolic–elliptic Keller–Segel aggregation equation arising in mathematical biology. In particular, we prove that there exists a global weak solution, if the order of the fractional diffusion $\alpha \in (1 - c_1, 2]$, where $c_1 > 0$ is an explicit constant depending on the physical parameters present in the problem (chemosensitivity and strength of logistic damping). Furthermore, in the range $1 - c_2 < \alpha \leq 2$ with $0 < c_2 < c_1$, the solution is globally smooth. Let us emphasize that when $\alpha < 1$, the diffusion is in the supercritical regime.

1. Introduction

The drift–diffusion equation

$$\partial_t u = -\nu \Lambda^\alpha u + \nabla \cdot (uB(u)) + f(u), \tag{1}$$

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where $B(u)$ is typically a vector of nonlocal operators and $\Lambda = \sqrt{-\Delta}$ (see (11) below), appears widely in applications. The parameter $0 \leq \alpha \leq 2$ is the order of the diffusion and it measures the strength of the viscous effects.

For instance, the two dimensional incompressible Navier–Stokes equations in its vorticity formulation can be written as

$$\partial_t u = \Delta u + \nabla \cdot (u \nabla^\perp \Delta^{-1} u), \tag{2}$$

where $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$. This equation governs the motion of two-dimensional, incompressible, homogeneous fluids in absence of forcing (see [48]). Equation (2) can be recovered from equation (1) by taking $\nu = 1$, $\alpha = 2$, $f = 0$ and $B = \nabla^\perp \Delta^{-1}$.

Another famous equation akin to (1) is the parabolic–elliptic simplification of the Keller–Segel system with logistic source

$$\partial_t u = \Delta u + \nabla \cdot (u \nabla \Delta^{-1} u) + \mu u - r u^2. \tag{3}$$

It appears as a model of *chemotaxis*, i.e. the proliferation and motion of cells (see the pioneer work of E. Keller & L. Segel [41] and the reviews by A. Blanchet [9] and Hillen & Painter [40], whose cell-kinetics model (M8) is the doubly-parabolic version of (3)). Here $u \geq 0$ is the density of cells. To obtain (3) from (1) one takes $\nu = 1$, $\alpha = 2$, $B = \nabla \Delta^{-1}$ and $f = \mu u - r u^2$. Equation (3) is also a model of gravitational collapse (see the works by Biler [5] and Ascasibar, Granero-Belinchón & Moreno [1]).

Furthermore, in one spatial dimension, the equation

$$\partial_t u = -\partial_x (u H u), \tag{4}$$

where H denotes the Hilbert transform (see (10) and (12) below), has been proposed as a model of the dynamics of a dislocation density u (see [31] and the work by Biler, Karch & Monneau [7]). Equation (4) appears also in a totally different context, namely as a one-dimensional model of the surface quasi-geostrophic equation (see Castro & Córdoba [16]). In order to recover (4) from (1), we choose $\nu = 0$, $f = 0$ and $B = -H$.

Finally, notice that the famous Burgers equation

$$\partial_t u = -\Lambda^\alpha u - u \partial_x u \tag{5}$$

can be obtained from (1) by taking $B(u) = -u/2$.

In all these equations there is a competition between the diffusion term given by

$$\Lambda^\alpha$$

and the transport term

$$\nabla \cdot (u B(u)).$$