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Chromatic quasisymmetric functions

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We introduce a quasisymmetric refinement of Stanley's chromatic symmetric function. We derive refinements of both Gasharov's Schur-basis expansion of the chromatic symmetric function and Chow's expansion in Gessel's basis of fundamental quasisymmetric functions. We present a conjectural refinement of Stanley's power sum basis expansion, which we prove in special cases. We describe connections between the chromatic quasisymmetric function and both the q-Eulerian polynomials introduced in our earlier work and, conjecturally, representations of symmetric groups on cohomology of regular semisimple Hessenberg varieties, which have been studied by Tymoczko and others. We discuss an approach, using the results and conjectures herein, to the e-positivity conjecture of Stanley and Stembridge for incomparability graphs of $(3+1)$-free posets.

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1. Introduction

We study a quasisymmetric refinement of Stanley's chromatic symmetric function. We present refined results for our quasisymmetric functions, some proved herein and some conjectured, analogous to known results and conjectures of Chow, Gasharov, Stanley and Stanley–Stembridge on chromatic symmetric functions. We present also a conjecture relating our work to work of Tymoczko and others on representations of symmetric groups on the cohomology of regular semisimple Hessenberg varieties. Some of the results in this paper were presented, without proof, in our survey paper [30]. We assume throughout that the reader is familiar with basic properties of symmetric and quasisymmetric functions, as discussed in [34, Chapter 7].

Let $G = (V, E)$ be a graph. Given a subset S of the set $\mathbb{P}$ of positive integers, a *proper S-coloring* of G is a function $\kappa : V \to S$ such that $\kappa(i) \neq \kappa(j)$ whenever $\{i, j\} \in E$. Let $\mathcal{C}(G)$ be the set of proper $\mathbb{P}$-colorings of G. In [32], Stanley defined the *chromatic symmetric function* of G as

$$X_G(\mathbf{x}) := \sum_{\kappa \in \mathcal{C}(G)} \mathbf{x}_\kappa,$$

where $\mathbf{x} := (x_1, x_2, \dots)$ is a sequence of commuting indeterminants and $\mathbf{x}_\kappa := \prod_{v \in V} x_{\kappa(v)}$.

It is straightforward to confirm that $X_G(\mathbf{x})$ lies in the $\mathbb{Q}$-algebra $\Lambda_{\mathbb{Q}}$ of symmetric functions in $x_1, x_2, \dots$ with rational coefficients. The chromatic symmetric function gives more information about proper colorings than the well-studied chromatic polynomial $\chi_G : \mathbb{P} \to \mathbb{N}$. (Recall that $\chi_G(m)$ is the number of proper $\{1, 2, \dots, m\}$-colorings of G.) Indeed, $X_G(1^m) = \chi_G(m)$, where $X_G(1^m)$ is the specialization of $X_G(\mathbf{x})$ obtained by setting $x_i = 1$ for $1 \le i \le m$ and $x_i = 0$ for $i > m$. Chromatic symmetric functions are studied in various papers, including [32,33,12,13,5,6,44,23,22,19,16].