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A Lie-algebraic approach to the local index theorem on compact homogeneous spaces

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A B S T R A C T

Using a K-theory point of view, Bott related the Atiyah–Singer index theorem for elliptic operators on compact homogeneous spaces to the Weyl character formula. This article explains how to prove the local index theorem for compact homogenous spaces using Lie algebra methods. The method follows in outline the proof of the local index theorem due to Berline and Vergne. But the use of Kostant's cubic Dirac operator in place of the Riemannian Dirac operator leads to substantial simplifications. An important role is also played by the quantum Weil algebra of Alekseev and Meinrenken.

1. Introduction

Soon after Atiyah and Singer [7] proved the index theorem, Bott [11] examined it over homogeneous spaces G/K where G is a compact connected Lie group and K is a closed connected subgroup. Using Weyl's theory—integral formula, character formula,

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and so on—Bott was able to verify the index theorem by-passing analytic or topological arguments considerably. Expectation of such simplification for the *local* index theorem is the motive of this article.

It seems that the well-known proofs for the local index theorem (such as the ones found in Atiyah, Bott, and Patodi [6], Getzler [17], Bismut [10], or Berline and Vergne [9]) in themselves do not become simpler even if we restrict the manifolds under consideration to compact homogeneous spaces. A common feature of the aforementioned proofs is that they use the Riemannian Dirac operator. What we shall demonstrate is that the use of Kostant's *cubic Dirac operator* [28] in place of the Riemannian Dirac operator leads to substantial simplifications; Alekseev and Meinrenken's *quantum Weil algebra* [2,3] also plays an important role.

This is certainly not the first instance where the utility of the cubic Dirac operator is found. Its usefulness has already been demonstrated in the realm of the representation theory of complex semisimple Lie algebras. We do not list all the works in that direction but refer to the treatise by Huang and Pandžić [22]. On a more differential geometric side, the advantage of the cubic Dirac operator for index theoretic purposes has been indicated by the works of Slebarski [31–33] and Goette [18]. More recently, Freed, Hopkins, and Teleman [16] used a family of cubic Dirac operators in the context of loop groups to prove the “Thom isomorphism” in that setting.

We shall outline our approach in the next few paragraphs. But first let us recapitulate the local index theorem. Let $\mathcal{D}$ be a Dirac operator on a vector bundle S over a closed even-dimensional manifold M. Owing to the representation theory of Clifford algebras, the space $\Gamma(S)$ of the smooth sections of S is naturally bi-graded:

$$\Gamma(S) = \Gamma(S)^+ \oplus \Gamma(S)^-.$$

In the most interesting cases the Dirac operator is an odd operator: $\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_- \\ \mathcal{D}_+ & 0 \end{pmatrix}$. It may be self-adjoint or skew self-adjoint, depending on the Clifford relation that defines the Clifford algebra. Because we will rely on the results of Alekseev and Meinrenken, we shall follow their convention, under which we have:

$$XY + YX = \langle X, Y \rangle \tag{1}$$

for vectors X and Y in the Euclidean space with inner product $\langle\ ,\ \rangle$ that generate the Clifford algebra. This makes $\mathcal{D}$ skew self-adjoint. Then the spectrum of $\mathcal{D}$ is an unbounded discrete subset of $i\mathbb{R}$, each eigenvalue occurring with finite multiplicity. The space $\Gamma^2(S)$ of square-integrable sections of S admits a Hilbert space direct sum decomposition into the eigenspaces of $\mathcal{D}$. (For general reference on the analytic properties of Dirac operators, see Roe [29, Chs. 5 & 7].) One can then build the heat diffusion operator $e^{t\mathcal{D}^2}$ on $\Gamma^2(S)$ for $t \in (0, \infty)$. The main reason for taking this step arises from the *McKean–Singer formula*, which relates the graded index of $\mathcal{D}$ with the super trace of the heat diffusion operator; namely,