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On discrete projective transformation groups of Riemannian manifolds

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ABSTRACT

We prove rigidity facts for groups acting on pseudo-Riemannian manifolds by preserving unparameterized geodesics.

RÉSUMÉ

Nous démontrons des résultats de rigidité pour les groupes agissant sur des variétés pseudo-riemanniennes en préservant leurs géodésiques non-paramétrées.

1. Introduction

1.0.1. The projective group of a connection

Two linear connections ∇ and ∇' on a manifold M are equal iff they have the same (parameterized) geodesics. They are called projectively equivalent if they have the same

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unparameterized geodesics. This is equivalent to that the difference $(2,1)$-tensor $T = \nabla - \nabla'$ being trace free in a natural sense [13].

The affine group $\mathsf{Aff}(M, \nabla)$ is that of transformations preserving ∇ and the projective one $\mathsf{Proj}(M, \nabla)$ is that of transformations f sending ∇ to a projectively equivalent one. So, elements of Aff are those preserving (parameterized) geodesics and those of Proj preserve unparameterized geodesics.

Obviously $\mathsf{Aff} \subset \mathsf{Proj}$; and it is natural to look for special connections for which this inclusion is proper, that is, when projective non-affine transformations exist.

1.0.2. Case of Levi-Civita connections

Let now g be a Riemannian metric on M and ∇ its Levi-Civita connection. The affine and projective groups $\mathsf{Aff}(M, g)$ and $\mathsf{Proj}(M, g)$ are those associated to ∇.

More generally, g and g' are projectively equivalent if so is the case for their associated Levi-Civita connections. This defines an equivalence relation on the space $\mathsf{Riem}(M)$ of Riemannian metrics on M. Let $\mathcal{P}(M, g)$ denote the class of g, i.e. the set of metrics shearing the same unparameterized geodesics with g. It contains $\mathbb{R}^+.g$, the set of constant multiples of g. Generically, $\mathcal{P}(M, g) = \mathbb{R}^+.g$.

One crucial fact here is that $\mathcal{P}(M, g)$ is always a finite dimensional manifold whose dimension is called the *degree of projective mobility* of g. (This contrasts with the case of projective equivalence classes of connections which are infinitely dimensional affine spaces. Similarly, conformal classes of metrics are identified to spaces of positive functions on the manifold.) It is actually one culminant fact of projective differential geometry to identify $\mathcal{P}(M, g)$ to an open subset of a finite dimensional linear sub-space $\mathcal{L}(M, g)$ of endomorphisms of TM (see §3). Being projectively equivalent for connections is a linear condition, but this is no longer linear for metrics (say because the correspondence $g \to$ its Levi-Civita connection, is far from being linear!). The trick is to perform a transform leading to a linear equation, see [6] for a nice exposition.

1.0.3. Philosophy

The idea behind our approach here is to let a diffeomorphism f on a differentiable manifold M act on the space $\mathsf{Riem}(M)$ of Riemannian metrics on M. That this action has a fixed point means exactly that f is an isometry for some Riemannian metric on M. One then naturally wonders what is the counterpart of the fact that the f-action preserves some (finite dimensional) manifold $V \subset \mathsf{Riem}(M)$. A classical similar idea is to let the isotopy class of a diffeomorphism on a surface act on its Teichmuller space [34]. Here, as it will be seen bellow, we are specially concerned with the case $\dim V = 2$.

1.0.4. More general pseudo-Riemannian framework

All this generalizes to the pseudo-Riemannian case. One fashion to unify all is to generalize all this to the wider framework of second order ordinary differential equations (e.g. Hamiltonian systems) on M, by letting their solutions play the role of (parameterized) geodesics.