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Closed characteristics on compact convex hypersurfaces in $\mathbf{R}^8$

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ABSTRACT

In this paper, we prove there exist at least four geometrically distinct closed characteristics on every compact convex hypersurface Σ in $\mathbf{R}^8$. This gives a confirmed answer in the case $n = 4$ to a long standing conjecture in Hamiltonian analysis since the time of A. M. Liapounov in 1892 (cf. P. 235 of [4]).

1. Introduction and main results

In this paper, let Σ be a C^3 compact convex hypersurface in $\mathbf{R}^{2n}$, i.e., Σ is the boundary of a compact and strictly convex region U in $\mathbf{R}^{2n}$. We denote the set of all such hypersurfaces by $\mathcal{H}(2n)$. Without loss of generality, we suppose that U contains the

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origin. We consider closed characteristics (τ, y) on Σ, which are solutions of the following problem

$$\begin{cases} \dot{y} = JN_\Sigma(y), \\ y(\tau) = y(0), \end{cases} \tag{1.1}$$

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ is the standard symplectic matrix in $\mathbf{R}^{2n}$, I_n is the identity matrix in $\mathbf{R}^n$, $\tau > 0$ is the period of y, $N_\Sigma(y)$ is the outward normal vector of Σ at y normalized by the condition $N_\Sigma(y) \cdot y = 1$. Here $a \cdot b$ denotes the standard inner product of $a, b \in \mathbf{R}^{2n}$. A closed characteristic (τ, y) is *prime*, if τ is the minimal period of y. Two closed characteristics (τ, y) and (σ, z) are *geometrically distinct*, if $y(\mathbf{R}) \neq z(\mathbf{R})$. We denote by $\mathcal{T}(\Sigma)$ the set of all geometrically distinct closed characteristics on Σ. A closed characteristic (τ, y) is *non-degenerate*, if 1 is a Floquet multiplier of y of precisely algebraic multiplicity 2, and is *elliptic*, if all the Floquet multipliers of y locate on $\mathbf{U} = \{z \in \mathbf{C} \,|\, |z| = 1\}$, i.e., the unit circle in the complex plane. It is *hyperbolic*, if 1 is a double Floquet multiplier of it and all the other Floquet multipliers of y are away from $\mathbf{U}$.

It is surprising enough that A. M. Liapounov in [14] of 1892 and J. Horn in [13] of 1903 were able to prove the following great result: *Suppose* $H : \mathbf{R}^{2n} \to \mathbf{R}$ *is analytic,* $\sigma(JH''(0)) = \{\pm\sqrt{-1}\omega_1, \ldots, \pm\sqrt{-1}\omega_n\}$ *are purly imaginary and satisfy* $\frac{\omega_i}{\omega_j} \notin \mathbf{Z}$ *for all* i, j. *Then there exists* $\epsilon_0 > 0$ *small enough such that*

$$^{\#}\mathcal{T}(H^{-1}(\epsilon)) \geq n, \qquad \forall\, 0 < \epsilon \leq \epsilon_0. \tag{1.2}$$

This deep result was greatly improved by A. Weinstein in [26] of 1973. He was able to prove that for $H \in C^2(\mathbf{R}^{2n}, \mathbf{R})$, if $H''(0)$ is positive definite, then there exists $\epsilon_0 > 0$ small such that (1.2) still holds. In [6], I. Ekeland and J. Lasry proved that if there exists $x_0 \in \mathbf{R}^{2n}$ such that

$$r \leq |x - x_0| \leq R, \qquad \forall x \in \Sigma$$

and $\frac{R}{r} < \sqrt{2}$, then $^{\#}\mathcal{T}(\Sigma) \geq n$.

Note that we have the following example of weakly non-resonant ellipsoid: Let $r = (r_1, \ldots, r_n)$ with $r_i > 0$ for $1 \leq i \leq n$. Define

$$\mathcal{E}_n(r) = \left\{ z = (x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbf{R}^{2n} \;\middle|\; \frac{1}{2}\sum_{i=1}^{n} \frac{x_i^2 + y_i^2}{r_i^2} = 1 \right\}$$

where $\frac{r_i}{r_j} \notin \mathbf{Q}$ whenever $i \neq j$. In this case, the corresponding Hamiltonian system is linear and all the solutions of (1.1) can be computed explicitly. Thus it is easy to verify that $^{\#}\mathcal{T}(\mathcal{E}_n(r)) = n$ and all the closed characteristics on $\mathcal{E}_n(r)$ are elliptic and