

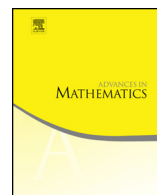


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A valuation-theoretic approach to translative-equidecomposability



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ABSTRACT

All simple translation-invariant valuations on polytopes are classified. As a direct consequence the well-known conditions for translative-equidecomposability are recovered. Furthermore, a simplified proof of the classification of *continuous* simple translation-invariant valuations is presented.

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1. Introduction

The study of equidecomposability has always been closely connected to valuation theory. In fact, Dehn's solution of Hilbert's Third Problem used valuations as the core ingredient.

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Let G be a subgroup of the group of motions that contains all translations. Two polytopes in \mathbb{R}^n are said to be G -equidecomposable if they can be cut into finitely many pieces such that there is a bijection between the two sets of pieces and corresponding pieces are equal up to a transformation from G .

A valuation ϕ is a map from the set of polytopes to \mathbb{R} such that

$$\phi(P \cup Q) = \phi(P) + \phi(Q) - \phi(P \cap Q)$$

for all polytopes P and Q whenever $P \cup Q$ is also convex.

After Dehn’s hallmark result a systematic study of valuations was initiated by Hadwiger [18] in the 1950s. In recent years the interest in valuations has increased tremendously (see e.g. [4–6,9,10,23,37,38,45]). Classification and characterization results have been a particular focus (see e.g. [1–3,7,8,12–16,20,25–31,35,36,42–44]).

One of the far reaching results of Hadwiger [17] (see also McMullen [33]) is a complete classification of weakly-continuous simple translation-invariant valuations. Here, \mathcal{U} denotes the set of (orthonormal) frames and \mathcal{U}_P^{n-k} denotes those frames that are P -tight and have $n - k$ entries. See Section 2 for precise definitions of the notation.

Theorem. (Cf. *Theorem 4.3.*) *A map $\phi: \mathcal{P}^n \rightarrow \mathbb{R}$ is a weakly-continuous simple translation-invariant valuation if and only if for all $U \in \mathcal{U}$ there exists a constant $c_U \in \mathbb{R}$ such that $U \mapsto c_U$ is odd and*

$$\phi(P) = \sum_{k=1}^n \sum_{U \in \mathcal{U}_P^{n-k}} c_U V_k(P_U)$$

for all $P \in \mathcal{P}^n$.

Our main result generalizes this classification to simple translation-invariant valuations without any regularity assumption.

Theorem. (See *Theorem 4.2.*) *A map $\phi: \mathcal{P}^n \rightarrow \mathbb{R}$ is a simple translation-invariant valuation if and only if for all $U \in \mathcal{U}$ there exists an additive function $f_U: \mathbb{R} \rightarrow \mathbb{R}$ such that $U \mapsto f_U$ is odd and*

$$\phi(P) = \sum_{k=1}^n \sum_{U \in \mathcal{U}_P^{n-k}} f_U(V_k(P_U))$$

for all $P \in \mathcal{P}^n$.

Hadwiger’s [18] *formal main criterion* (in German: *Formales Hauptkriterium*) establishes a connection between the G -equidecomposability of two polytopes and simple G -invariant valuations. It states that two polytopes are G -equidecomposable if and only if their values agree for every simple G -invariant valuation. Hence, it is possible to solve

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