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A valuation-theoretic approach to translative-equidecomposability

Katharina Kusejko, Lukas Parapatits *

ETH Zurich, Department of Mathematics, Rämistrasse 101, 8092 Zürich, Switzerland

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All simple translation-invariant valuations on polytopes are classified. As a direct consequence the well-known conditions for translative-equidecomposability are recovered. Furthermore, a simplified proof of the classification of *continuous* simple translation-invariant valuations is presented.

1. Introduction

The study of equidecomposability has always been closely connected to valuation theory. In fact, Dehn's solution of Hilbert's Third Problem used valuations as the core ingredient.

* Corresponding author.
E-mail addresses: katharina.kusejko@math.ethz.ch (K. Kusejko), lukas.parapatits@math.ethz.ch (L. Parapatits).

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Let G be a subgroup of the group of motions that contains all translations. Two polytopes in $\mathbb{R}^n$ are said to be G-equidecomposable if they can be cut into finitely many pieces such that there is a bijection between the two sets of pieces and corresponding pieces are equal up to a transformation from G.

A valuation ϕ is a map from the set of polytopes to $\mathbb{R}$ such that

$$\phi(P \cup Q) = \phi(P) + \phi(Q) - \phi(P \cap Q)$$

for all polytopes P and Q whenever $P \cup Q$ is also convex.

After Dehn's hallmark result a systematic study of valuations was initiated by Hadwiger [18] in the 1950s. In recent years the interest in valuations has increased tremendously (see e.g. [4–6,9,10,23,37,38,45]). Classification and characterization results have been a particular focus (see e.g. [1–3,7,8,12–16,20,25–31,35,36,42–44]).

One of the far reaching results of Hadwiger [17] (see also McMullen [33]) is a complete classification of weakly-continuous simple translation-invariant valuations. Here, $\mathcal{U}$ denotes the set of (orthonormal) frames and $\mathcal{U}_P^{n-k}$ denotes those frames that are P-tight and have $n-k$ entries. See Section 2 for precise definitions of the notation.

Theorem. *(Cf. Theorem 4.3.) A map $\phi: \mathcal{P}^n \to \mathbb{R}$ is a weakly-continuous simple translation-invariant valuation if and only if for all $U \in \mathcal{U}$ there exists a constant $c_U \in \mathbb{R}$ such that $U \mapsto c_U$ is odd and*

$$\phi(P) = \sum_{k=1}^{n} \sum_{U \in \mathcal{U}_P^{n-k}} c_U V_k(P_U)$$

for all $P \in \mathcal{P}^n$.

Our main result generalizes this classification to simple translation-invariant valuations without any regularity assumption.

Theorem. *(See Theorem 4.2.) A map $\phi: \mathcal{P}^n \to \mathbb{R}$ is a simple translation-invariant valuation if and only if for all $U \in \mathcal{U}$ there exists an additive function $f_U: \mathbb{R} \to \mathbb{R}$ such that $U \mapsto f_U$ is odd and*

$$\phi(P) = \sum_{k=1}^{n} \sum_{U \in \mathcal{U}_P^{n-k}} f_U(V_k(P_U))$$

for all $P \in \mathcal{P}^n$.

Hadwiger's [18] *formal main criterion* (in German: *Formales Hauptkriterium*) establishes a connection between the G-equidecomposability of two polytopes and simple G-invariant valuations. It states that two polytopes are G-equidecomposable if and only if their values agree for every simple G-invariant valuation. Hence, it is possible to solve