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Classification of positive $\mathcal{D}^{1,p}(\mathbb{R}^N)$-solutions to the critical p-Laplace equation in $\mathbb{R}^N$ ☆

Berardino Sciunzi

Dipartimento di Matematica e Informatica, Università della Calabria, Ponte Pietro Bucci 31B, I-87036 Arcavacata di Rende, Cosenza, Italy

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ABSTRACT

We provide the classification of the positive solutions to $-\Delta_p u = u^{p^*-1}$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ in the case $2 < p < N$. Since the case $1 < p \leq 2$ is already known this provides the complete classification for $1 < p < N$.

Contents

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E-mail address: sciunzi@mat.unical.it.

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1. Introduction

We consider in the whole space the critical problem

$$\mathcal{P}^* := \begin{cases} -\Delta_p u = u^{p^*-1} & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N \\ u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \end{cases}$$

where $1 < p < N$, $p^* = \frac{Np}{N-p}$ is the critical exponent for the Sobolev embedding and

$$\mathcal{D}^{1,p}(\mathbb{R}^N) = \Big\{ u \in L^{p^*}(\mathbb{R}^N) \, : \, \int_{\mathbb{R}^N} |\nabla u|^p < \infty \Big\}.$$

Let us recall that any solution $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ of $\mathcal{P}^*$ belongs to $L^\infty(\mathbb{R}^N)$ as it follows by [15,17,23]. Consequently we have that u is locally of class $C^{1,\alpha}$ by $C^{1,\alpha}$ estimates (see [9,13,14,21,22]).

We deal with the classification of the solutions to $\mathcal{P}^*$. It is well known that such issue is crucial in many applications such as a priori estimates, blow-up analysis and asymptotic analysis. An explicit family of solutions to $\mathcal{P}^*$ is given by

$$U_{\lambda,x_0} := \left[\frac{\lambda^{\frac{1}{p-1}} \left(N^{\frac{1}{p}} \left(\frac{N-p}{p-1} \right)^{\frac{p-1}{p}} \right)}{\lambda^{\frac{p}{p-1}} + |x - x_0|^{\frac{p}{p-1}}} \right]^{\frac{N-p}{p}} \qquad \lambda > 0 \qquad x_0 \in \mathbb{R}^N. \tag{1.1}$$

Note that, by [20], it follows that the family of functions given by (1.1) are minimizers to

$$S := \min_{\substack{\varphi \in \mathcal{D}^{1,p}(\mathbb{R}^N) \\ \varphi \neq 0}} \frac{\int_{\mathbb{R}^N} |\nabla \varphi|^p dx}{\left(\int_{\mathbb{R}^N} \varphi^{p^*} dx \right)^{\frac{p}{p^*}}}. \tag{1.2}$$

By the classification results in [12] (see also [1]) it follows that all the regular *radial* solutions to $\mathcal{P}^*$ are given by (1.1).

In the semilinear case $p = 2$ it has been proved in the celebrated paper [2] (see also [3]) that any solution to $-\Delta u = u^{\frac{N+2}{N-2}}$ $(N \geq 3)$ is radial and hence classified by (1.1). It is crucial in the proof the use of the Kelvin transform that allows to reduce to the study of the symmetry of solutions that have nice decaying properties at infinity. Previous results were proved in [11] via the *Moving Plane Method* under additional conditions, see [18] and [10].

In the quasilinear case $p \neq 2$ the problem is more difficult and we have to take into account the nonlinear nature of the p-Laplace operator, the lack of regularity of the solutions and the fact that comparison principles are not equivalent to maximum principles in this case. Furthermore a Kelvin type transform is not available. The first