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Geometry of the arithmetic site

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ABSTRACT

We introduce the *Arithmetic Site*: an algebraic geometric space deeply related to the noncommutative geometric approach to the Riemann Hypothesis. We prove that the noncommutative space quotient of the adèle class space of the field of rational numbers by the maximal compact subgroup of the idèle class group, which we had previously shown to yield the correct counting function to obtain the complete Riemann zeta function as Hasse–Weil zeta function, is the set of geometric points of the arithmetic site over the semifield of tropical real numbers. The action of the multiplicative group of positive real numbers on the adèle class space corresponds to the action of the Frobenius automorphisms on the above geometric points. The underlying topological space of the arithmetic site is the topos of functors from the multiplicative semigroup of non-zero natural numbers to the category of sets. The structure sheaf is made by semirings of characteristic one and is given globally by the semifield of tropical integers. In spite of the countable combinatorial nature of the arithmetic site, this space admits a one parameter semigroup of Frobenius correspondences obtained as sub-varieties of the square of the site. This square is a semi-ringed topos whose structure sheaf involves Newton polygons. Finally, we show

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that the arithmetic site is intimately related to the structure of the (absolute) point in non-commutative geometry.

1. Introduction

It has long been known since [5] that the noncommutative space of adèle classes of a global field provides a framework to interpret the explicit formulas of Riemann–Weil in number theory as a trace formula. In [6], we showed that if one divides the adèle class space $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times$ of the rational numbers by the maximal compact subgroup $\hat{\mathbb{Z}}^\times$ of the idèle class group, one obtains by considering the induced action of $\mathbb{R}_+^\times$, the counting distribution $N(u)$, $u \in [1, \infty)$, which determines, using the Hasse–Weil formula in the limit $q \to 1$, the complete Riemann zeta function. This analytic construction provides the starting point of the noncommutative attack to the Riemann Hypothesis. In order to adapt the geometric proof of A. Weil, what was still missing until now was the definition of a geometric space of classical type whose points (defined over a "field" replacing the algebraic closure of the finite field $\mathbb{F}_q$ as $q \to 1$) would coincide with the afore mentioned quotient space. The expectation being that the action of suitably defined Frobenius automorphisms on these points would correspond to the above action of $\mathbb{R}_+^\times$.

The primary intent of this paper is to provide a natural solution to this search by introducing (*cf.* Definition 3.1) the *arithmetic site* as an object of algebraic geometry involving two elaborate mathematical concepts: the notion of topos and of (structures of) characteristic 1 in algebra. The topological space underlying the arithmetic site is the Grothendieck topos of sets with an action of the multiplicative monoïd $\mathbb{N}^\times$ of non-zero positive integers. The structure sheaf of the arithmetic site is a fundamental semiring of characteristic 1, *i.e.* $\mathbb{Z}_{\max} := (\mathbb{Z} \cup \{-\infty\}, \max, +)$ on which $\mathbb{N}^\times$ acts by Frobenius endomorphisms. The role of the algebraic closure of $\mathbb{F}_q$, in the limit $q \to 1$, is provided by the semifield $\mathbb{R}_+^{\max}$ of tropical real numbers which is endowed with a one parameter group of Frobenius automorphisms Fr_λ, $\lambda \in \mathbb{R}_+^\times$, given by $\mathrm{Fr}_\lambda(x) = x^\lambda \ \forall x \in \mathbb{R}_+^{\max}$.

In this article we prove the following

Theorem 1.1. *The set of points of the arithmetic site* $(\widehat{\mathbb{N}^\times}, \mathbb{Z}_{\max})$ *over* $\mathbb{R}_+^{\max}$ *coincides with the quotient of* $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times$ *by the action of* $\hat{\mathbb{Z}}^\times$. *The action of the Frobenius automorphisms* Fr_λ *of* $\mathbb{R}_+^{\max}$ *on these points corresponds to the action of the idèle class group on* $\hat{\mathbb{Z}}^\times \backslash \mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times$.

The definition of the arithmetic site arises as a natural development of our recent work which underlined the following facts

– The theory of toposes of Grothendieck provides the best geometric framework to understand cyclic (co)homology and the λ-operations using the (presheaf) topos associated to the cyclic category [3] and its epicyclic refinement (*cf.* [8]).