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Non-formal star-exponential on contracted one-sheeted hyperboloids

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A B S T R A C T

In this paper, we exhibit the non-formal star-exponential of the Lie group $SL(2,\mathbb{R})$ realized geometrically on the curvature contraction of its one-sheeted hyperboloid orbits endowed with its natural non-formal star-product. It is done by a direct resolution of the defining equation of the star-exponential and produces an expression with Bessel functions. This yields a continuous group homomorphism from $SL(2,\mathbb{R})$ into the von Neumann algebra of multipliers of the Hilbert algebra associated to this natural star-product. As an application, we prove a new identity on Bessel functions.

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1. Introduction

Deformation quantization, initiated in [7], consists in deforming the pointwise product of the commutative algebra of smooth functions $\mathcal{C}^\infty(M)$ on a Poisson manifold M into a noncommutative star-product $\star_\theta$ depending on a deformation parameter θ. Formal deformation quantizations were intensively studied [32,30,22,26] and definitely classified in [27]. In the non-formal setting, there exist some examples of deformation of groups and their actions like Abelian Lie groups $\mathbb{R}^{2n}$ [35], Abelian Lie supergroups [14,19], Kählerian Lie groups [11], Abelian p-adic groups [25], deformations of $\mathbb{C}^{2n}$ in holomorphic [33,12] or resurgent [24] context, deformations of $SU(1,n)$ [13,28], but no general classifying theory is available.

Associated to star-products and following [23], the notion of star-exponential [7,6,1] plays an important role in the study of deformation quantization, for giving access to spectrum of operators [17,18], for the link with representation theory. In the non-formal context, star-exponentials of quadratic functions were explicitly computed [33,34] for the Moyal–Weyl product. Applications to harmonic analysis such as character formula or Fourier transformation can be obtained by computing non-formal star-exponential of momentum maps of some Lie group's action. This was performed for nilpotent Lie groups in [2] by using the Moyal–Weyl product. By using Berezin and Weyl quantizations, this program of star-representations was achieved in the case of unitary irreducible representations of compact semisimple Lie groups [4], of holomorphic discrete series [5] (see also [8] for $SL_2(\mathbb{R})$) and principal series of semisimple Lie groups [16] (see also [3]).

However, one can wonder whether it is possible to construct non-formal star-exponentials for star-products that are geometrically more natural for the orbits of the group. In this spirit, the non-formal star-exponential of Kählerian Lie groups with negative curvature was exhibited in [15] for invariant star-products on their coadjoint orbits, with application to the construction of an adapted Fourier transformation.

In this paper, we are interested in the one-sheeted hyperboloid orbits of $SL_2(\mathbb{R})$ [29], also called two-dimensional anti-de Sitter space, $AdS_2 := SL_2(\mathbb{R})/SO(1,1)$. To compute its star-exponential, we want to dispose of a non-formal $\mathfrak{sl}_2(\mathbb{R})$-covariant star-product geometrically adapted to AdS_2, and to this aim, we will look at its natural contraction. Let us first show that this contraction of AdS_2 corresponds locally to the symmetric space $M := SO(1,1) \ltimes \mathbb{R}^2/\mathbb{R}$ called Poincaré coset. The global picture is however given in the conclusion of this paper but it is not needed now.

This curvature contraction is induced by the contraction of Lie algebras:

$$\mathfrak{sl}_2(\mathbb{R}) \longrightarrow \mathfrak{so}(1,1) \ltimes \mathbb{R}^2$$

that corresponds to the limit $t \to 0$ in the following three-dimensional real Lie algebra $\mathfrak{g}_t$ table:

$$[H,E] = 2E, \qquad [H,F] = -2F, \qquad [E,F] = t\,H\ ,$$

where $\mathfrak{g}_t$ is isomorphic to $\mathfrak{g}_1 \simeq \mathfrak{sl}_2(\mathbb{R})$ for every $t > 0$, while $\mathfrak{g}_0 \simeq \mathfrak{so}(1,1) \ltimes \mathbb{R}^2$.