Advances in Mathematics 291 (2016) 542–583

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Non-uniqueness of admissible weak solutions to compressible Euler systems with source terms

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ARTICLE INFO

Article history:
Received 4 July 2015
Received in revised form 7 December 2015
Accepted 31 December 2015
Available online 4 February 2016
Communicated by Camillo De Lellis

Keywords:
Non-uniqueness
Admissible weak solutions
Finite states
Source terms
Rotating
Damping

ABSTRACT

We consider admissible weak solutions to the compressible Euler system with source terms, which include rotating shallow water system and the Euler system with damping as special examples. In the case of anti-symmetric sources such as rotations, for general piecewise Lipschitz initial densities and some suitably constructed initial momentum, we obtain infinitely many global admissible weak solutions. Furthermore, we construct a class of finite-states admissible weak solutions to the Euler system with anti-symmetric sources. Under the additional smallness assumption on the initial densities, we also obtain multiple global-in-time admissible weak solutions for more general sources including damping. The basic framework are based on the convex integration method developed by De Lellis and Székelyhidi [13,14] for the Euler system. One of the main ingredients of this paper is the construction of specified localized plane wave perturbations which are compatible with a given source term.

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http://dx.doi.org/10.1016/j.aim.2015.12.027
0001-8708/

1. Introduction

The well-posedness of the compressible Euler system is one of the central issues in hyperbolic balance laws. The uniqueness of admissible solutions is of fundamental importance. Although there is a quite mature well-posedness theory for one dimensional hyperbolic conservation laws with small BV initial data [12], the problem for multi-dimensional systems is very challenging. Recently, a major breakthrough for the uniqueness problem is made by De Lellis and Székelyhidi in [13,14]. Inspired by the surprising examples in [27] and [28], they developed a convex integration framework and obtained infinitely many bounded weak solutions to the incompressible Euler system. The convex integration methods are later refined to generate even Hölder continuous solutions for the incompressible Euler system, see [2,16,22]. The ideas have also been applied to other systems of PDEs; see [10,23,29,33] and the references therein. We refer to [15] for a general survey on the results in this direction.

Multiple admissible solutions to the compressible Euler system are obtained in [5,7,6,14,18], by an adaptation of the convex integration method. A major consequence is that the admissible weak solutions for the polytropic gases in multidimension are in general non-unique. These solutions, which are called "wild solutions" in the literature, reflect the flexibility of the solution space with low regularity and are quite different in nature from those in one dimensional setting such as [31]. It has been shown that many of the available criteria [7,14], with the exception of vanishing viscosity limit, are not able to single out a unique solution.

It is interesting to investigate the stability mechanisms which may help to rule out the wild solutions. A natural candidate is the lower order dissipation and dispersion, such as damping and rotating forces.

In this paper, we consider the compressible Euler system with source terms as follows

$$\begin{cases}\partial_t\rho + \nabla\cdot(\rho\mathbf{u}) = 0,\\ \partial_t(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p(\rho) = \mathbf{B}(\rho\mathbf{u}),\end{cases} \tag{1}$$

where $(x,t)\in\Omega\times[0,\infty)$ with $\Omega=\mathbb{R}^n$ or $\mathbb{T}^n$; ρ, $\mathbf{u}$, and p denote the density, velocity, and the pressure of the flows, respectively. Assume that the equation of states satisfies $p(0)=0$ and $p'(\rho)>0$ for $\rho>0$, and $\mathbf{B}$ is an $n\times n$ constant matrix. In particular, the effects of damping and rotating forces are included in this model, where in the case of $n=2$ and non-dimensionalization, the matrix $\mathbf{B}$ has the form $\mathbf{B}=-\mathbf{I}$ and $\mathbf{B}=\mathbf{J}$, respectively, with

$$\mathbf{I}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad\text{and}\quad \mathbf{J}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{2}$$

Most of the previous investigations [5,7,6,14,18] focused on the isentropic Euler system corresponding to $\mathbf{B}=\mathbf{0}$ in the system (1). The existence of infinitely many bounded