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Topological Galois theory

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ABSTRACT

We introduce an abstract topos-theoretic framework for building Galois-type theories in a variety of different mathematical contexts; such theories are obtained from representations of certain atomic two-valued toposes as toposes of continuous actions of a topological group. Our framework extends Grothendieck's theory of Galois categories and allows to build Galois-type equivalences in new contexts, such as for example graph theory and finite group theory.

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1. Introduction

The present work provides a general framework, based on Topos Theory, for building Galois-type theories in a variety of different mathematical contexts.

Most notably, we identify a set of necessary and sufficient conditions on a category (resp. on a small category) for it to be equivalent to the category of continuous actions (resp. of continuous non-empty transitive actions) of a topological group on discrete sets. We also intrinsically characterize the categories which can be represented as full subcategories of categories of non-empty transitive actions of a topological group, and describe an elementary process for 'completing' them so as to make them equivalent to such categories of actions.

We show in particular that many classical categories can be naturally embedded into Galois-type categories; for instance, this is the case for the category of finite linear orders and embeddings, the category of finite graphs and embeddings, the category of finite Boolean algebras and injective homomorphisms, or the category of finite groups and injective homomorphisms.

In order to illustrate our main results, we briefly review the classical (infinite) Galois theory and its categorical interpretation.

Let $F \subseteq L$ be a Galois extension, not necessarily finite-dimensional. The group $Aut_F(L)$ of automorphisms of L which fix F can be naturally made into a topological group by endowing it with the so-called *Krull topology*, that is the topology in which