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Order complexes of coset posets of finite groups are not contractible

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ABSTRACT

We show that the order complex of the poset of all cosets of all proper subgroups of a finite group G is never $\mathbb{F}_2$-acyclic and therefore never contractible. This settles a question of K.S. Brown.

1. Introduction

We settle a question asked by K.S. Brown in [9]. For a group G, $\mathcal{C}(G)$ will denote the poset of all cosets of all proper subgroups of G, ordered by inclusion. For a poset P, ΔP will denote the order complex of P. Other terms used but not defined in this introduction are defined in Section 2.

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Theorem 1.1. *If G is a finite group, then $\Delta\mathcal{C}(G)$ is not $\mathbb{F}_2$-acyclic, and therefore is not contractible.*

With some explicitly stated exceptions, the groups, partially ordered sets and simplicial complexes considered herein are assumed to be finite. We assume some familiarity with topological combinatorics (see for example [4,45]), along with the rudiments of algebraic topology (see for example [19,27]) and group theory (see for example [2,14]).

1.1. History and motivation

The topology of $\Delta\mathcal{C}(G)$ was studied by Brown in [9]. More general coset complexes were studied from a somewhat different point of view by Abels and Holz in [1]. However, from our perspective (and that of Brown), the story begins with the work of P. Hall, who in [18] introduced generalized Möbius inversion in order to enumerate generating sequences. Hall considered the probability $P_G(k)$ that a k-tuple $(g_1, \ldots, g_k)$ of elements of a group G, chosen uniformly with replacement, includes a generating set for G. He showed that

$$P_G(k) = \sum_{H \leq G} \mu(H, G)[G : H]^{-k},$$

where μ is the Möbius function on the subgroup lattice of G. (We mention that Weisner introduced generalized Möbius inversion independently in [46]. See [42, Chapter 3] for a comprehensive discussion of this theory.)

Bouc observed that $-P_G(-1)$ is the reduced Euler characteristic $\tilde{\chi}(\Delta\mathcal{C}(G))$. Indeed, Hall showed in [18] that if $\widehat{P}$ is obtained from P by adding a minimum element $\hat{0}$ and a maximum element $\hat{1}$, then

$$\tilde{\chi}(\Delta P) = \mu_{\widehat{P}}(\hat{0}, \hat{1}).$$

A straightforward computation shows that

$$\mu_{\widehat{\mathcal{C}(G)}}(\hat{0}, \hat{1}) = -P_G(-1).$$

This led to Brown's work, in which he obtained divisibility results for $P_G(-1)$ using group actions on $\Delta\mathcal{C}(G)$.

Brown found no group G for which $P_G(-1) = 0$. As the reduced Euler characteristic of a contractible complex is zero, the question of contractibility arises naturally. Previous progress on this question involved showing that $P_G(-1) \neq 0$. Gaschütz showed in [16, Satz 2] that $P_G(-1) \neq 0$ when G is solvable. (Brown refined this result by calculating the homotopy type of $\Delta\mathcal{C}(G)$ for a solvable group G in [9, Proposition 11].) Patassini proved $P_G(-1) \neq 0$ for many almost simple groups G in [29,30]. He obtained further results for some groups with minimal normal subgroups that are products of alternating