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Frankel conjecture and Sasaki geometry

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We classify simply connected compact Sasaki manifolds of dimension $2n + 1$ with positive transverse bisectional curvature. In particular, the Kähler cone corresponding to such manifolds must be bi-holomorphic to $\mathbb{C}^{n+1}\backslash\{0\}$. As an application we recover the theorem of Mori and Siu–Yau on the Frankel conjecture and extend it to certain orbifold version. The main idea is to deform such Sasaki manifolds to the standard round sphere in two steps, both fixing the complex structure on the Kähler cone. First, we deform the metric along the Sasaki–Ricci flow and obtain a limit Sasaki–Ricci soliton with positive transverse bisectional curvature. Then by varying the Reeb vector field which essentially decreases the volume functional, we deform the Sasaki–Ricci soliton to a Sasaki–Einstein metric with positive transverse bisectional curvature, i.e. a round sphere. The second deformation is only possible when one treats simultaneously regular and irregular Sasaki manifolds, even if the manifold one starts with is regular (quasi-regular), i.e. Kähler manifolds (orbifolds).

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1. Introduction and main results

In this paper we study compact Sasaki manifolds with positive transverse bisectional curvature. Sasaki geometry, in particular, Sasaki–Einstein manifolds have been studied extensively. Readers are referred to the monograph [5], the survey paper [50] and the references therein for the history and recent progress on this subject.

The study of manifolds with positive curvature is one of the most important subjects in Riemannian geometry. There is a lot of recent deep progress on this, especially using the technique of Ricci flow, see [4] and [9] for example. In Kähler geometry a natural concept is the positivity of the bisectional curvature. It was conjectured by Frankel [21] that a compact Kähler manifold of complex dimension n with positive bisectional curvature is biholomorphic to the complex projective space $\mathbb{CP}^n$. The Frankel conjecture was proved in later 1970s independently by Mori [42] (he proved the more general Hartshorne conjecture) via algebraic geometry and Siu–Yau [48] via differential geometry. Sasaki geometry is an odd dimensional companion of Kähler geometry, so it is very natural to ask for the counterpart of the theorem of Mori and Siu–Yau on the Frankel conjecture for Sasaki manifolds. This is the major point of study in this article. We would like to emphasize that this generalization seems to be interesting in that it provides a uniform framework which also proves the original Frankel conjecture, by deformation to canonical metrics, as attempted previously by many people (cf. [15,16,44]). Moreover, the use of Sasaki geometry also yields certain orbifold version of the Frankel conjecture, which seems to be difficult to obtain with the known approaches. Finally, as already pointed out in [4] a *pinching towards constant curvature* proof of the Frankel conjecture using Ricci flow seems not plausible, as there are examples of two dimensional Ricci soliton orbifolds with positive curvature. One of the applications of the results developed in this article is to classify such solitons, in a uniform way.

Sasaki geometry in dimension $2n + 1$ is closely related to Kähler geometry in both dimensions $2(n + 1)$ and $2n$. A Sasaki manifold M of dimension $2n + 1$ admits, on one hand, a Kähler cone structure on the product $X = M \times \mathbb{R}_+$, and on the other hand, a transverse Kähler structure on the (local) quotient by the Reeb vector field. For now we view a Sasaki structure on M as a Kähler cone structure on X, and we identify M with the link $\{r = 1\}$ in X. A standard example of a Sasaki manifold is the odd dimensional round sphere S^{2n+1}. The corresponding Kähler cone is $\mathbb{C}^{n+1}\backslash\{0\}$ with the flat metric.

A Sasaki manifold admits a canonical Killing vector field ξ, called the *Reeb vector field*. It is given by rotating the homothetic vector field $r\partial_r$ on X by the complex structure J. The integral curves of ξ are geodesics, and give rise to a foliation on M, called the *Reeb foliation*. Then there is a Kähler structure on the local leaf space of the Reeb foliation, called the *transverse Kähler structure*. If the transverse Kähler structure has positive bisectional curvature, we say the Sasaki manifold has *positive transverse bisectional curvature*. If the Sasaki manifold has positive sectional curvature, it automatically has positive transverse bisectional curvature, for example, the round metric on S^{2n+1}.