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Convergence in L^p for Feynman path integrals

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ABSTRACT

We consider a class of Schrödinger equations with time-dependent smooth magnetic and electric potentials having a growth at infinity at most linear and quadratic, respectively. We study the convergence in L^p with loss of derivatives, $1 < p < \infty$, of the time slicing approximations of the corresponding Feynman path integral. The results are completely sharp and hold for long time, where no smoothing effect is available. The techniques are based on the decomposition and reconstruction of functions and operators with respect to certain wave packets in phase space.

1. Introduction

Feynman path integrals were introduced in 1948 [20,21] to provide a new formulation of Quantum Mechanics and nowadays represent a fundamental tool in most branches of modern Physics. In particular, R. Feynman suggested the construction of the integral

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kernel $K(t,s,x,y)$ of the Schrödinger propagator as a suggestive sum-over-histories, in the following sense. First of all the kernel $K(t,s,x,y)$ itself is interpreted as the probability amplitude for a particle to be at the point x at time t provided it was at y at time s $(x,y \in \mathbb{R}^d)$. Now, in the computation of this quantity every path γ joining y and x, therefore satisfying $\gamma(s) = y$, $\gamma(t) = x$, carries a contribution which is proportional to $e^{i\hbar^{-1}S[\gamma]}$, where $S[\gamma]$ is the action along the path γ:

$$S[\gamma] = \int_s^t \mathcal{L}(\gamma(\tau), \dot{\gamma}(\tau), \tau)\, d\tau,$$

$\mathcal{L}$ being the Lagrangian of the corresponding classical system. The total amplitude is finally obtained by superposition and can be written symbolically as an integral

$$K(t,s,x,y) = \int e^{i\hbar^{-1}S[\gamma]}\mathcal{D}[\gamma]$$

over the space of paths satisfying the above boundary conditions. Although a suitable measure on this space does not exist in the measure theoretic sense (cf. [7]), several rigorous justifications have been proposed by many authors and from different viewpoints (analytic continuation of the parabolic propagator, infinite dimensional oscillatory integrals, stochastic integrals, etc.). The literature is enormous and we refer to the books [1,49,51,52] and the references therein. Instead here we focus on the original approach by Feynman [20,21] via time slicing approximations, which was carried on in a rigorous way in the papers [22,23,26–30,32,33,38,39,43–47,58] (see also [24,25,56]). Briefly one argues as follows. Suppose that for $|t-s|$ small enough there is only one classical path γ (i.e. a path satisfying the Euler–Lagrange equation) satisfying the boundary condition $\gamma(s) = y$, $\gamma(t) = x$. Define then the action

$$S(t,s,x,y) = \int_s^t \mathcal{L}(\gamma(\tau), \dot{\gamma}(\tau), \tau)\, d\tau, \tag{1}$$

along that path.

Consider the operator $E^{(0)}(t,s)$ defined by

$$E^{(0)}(t,s)f(x) = \frac{1}{(2\pi i(t-s)\hbar)^{d/2}} \int_{\mathbb{R}^d} e^{i\hbar^{-1}S(t,s,x,y)} f(y)\, dy. \tag{2}$$

The idea is that this operator should represent a good approximation of the actual propagator when $|t-s|$ is small (in fact, for the free particle $E^{(0)}(t,s)$ coincides with the exact propagator). In general one then considers a subdivision $\Omega : s = t_0 < t_1 < \ldots < t_L = t$ of the interval $[s,t]$ and the composition

$$E^{(0)}(\Omega,t,s) = E^{(0)}(t,t_{L-1})E^{(0)}(t_{L-1},t_{L-2}) \ldots E^{(0)}(t_1,s), \tag{3}$$