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Zonotopal algebra and forward exchange matroids

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A B S T R A C T

Zonotopal algebra is the study of a family of pairs of dual vector spaces of multivariate polynomials that can be associated with a list of vectors X. It connects objects from combinatorics, geometry, and approximation theory. The origin of zonotopal algebra is the pair $(\mathcal{D}(X), \mathcal{P}(X))$, where $\mathcal{D}(X)$ denotes the Dahmen–Micchelli space that is spanned by the local pieces of the box spline and $\mathcal{P}(X)$ is a space spanned by products of linear forms.
The first main result of this paper is the construction of a canonical basis for $\mathcal{D}(X)$. We show that it is dual to the canonical basis for $\mathcal{P}(X)$ that is already known.
The second main result of this paper is the construction of a new family of zonotopal spaces that is far more general than the ones that were recently studied by Ardila–Postnikov, Holtz–Ron, Holtz–Ron–Xu, Li–Ron, and others. We call the underlying combinatorial structure of those spaces forward exchange matroid. A forward exchange matroid is an ordered matroid together with a subset of its set of bases that satisfies a weak version of the basis exchange axiom.

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1. Introduction

A finite list of vectors X gives rise to a large number of objects in combinatorics, algebraic and discrete geometry, commutative algebra, and approximation theory. Examples include matroids, hyperplane arrangements and zonotopes, fat point ideals, and box splines. In the 1980s, various authors in the approximation theory community started studying algebraic structures that capture information about splines (*e.g.* [1,16,27]). One important example is the Dahmen–Micchelli space $\mathcal{D}(X)$, that is spanned by the local pieces of the box spline and their partial derivatives. See [34, Section 1.2] for a historic survey and the book [17] for a treatment of polynomial spaces appearing in the theory of box splines. Related results were obtained independently by authors interested in hyperplane arrangements (*e.g.* [45]).

The space $\mathcal{P}(X)$ that is dual to $\mathcal{D}(X)$ was introduced in [1,27]. It is spanned by products of linear forms and it can be written as the Macaulay inverse system (or kernel) of an ideal generated by powers of linear forms [15]. Ideals of this type and their inverse systems are also studied in the literature on fat point ideals [30,31] and graph orientations [4].

In addition to the aforementioned pair of spaces $(\mathcal{D}(X), \mathcal{P}(X))$, Olga Holtz and Amos Ron introduced two more pairs of spaces with interesting combinatorial properties [34]. They named the theory of those spaces *Zonotopal Algebra*. This name reflects the fact that there are various connections between zonotopal spaces and the lattice points in the zonotope defined by X if the list X is unimodular.

Subsequently, those results were further generalised by Olga Holtz, Amos Ron, and Zhiqiang Xu [35] as well as Nan Li and Amos Ron [42]. Federico Ardila and Alex Postnikov studied generalised $\mathcal{P}$-spaces and connections with power ideals [2]. Bernd Sturmfels and Zhiqiang Xu established a connection with Cox rings [51]. Further work on spaces of $\mathcal{P}$-type includes [5,8,39,41,52].

Zonotopal algebra is closely related to matroid theory: the Hilbert series of zonotopal spaces only depend on the matroid structure of the list X.

It is known that there is a canonical way to construct bases for the spaces of $\mathcal{P}$-type [2,27,34,40,42]. The first of the two main results in this paper is an algorithm that constructs a basis for spaces of $\mathcal{D}$-type. Two different algorithms are already known [13, 19]. However, our algorithm has several advantages over the other two: it is canonical and it yields a basis that is dual to the known basis for the $\mathcal{P}$-space. Here, canonical means that the basis that we obtain only depends on the order of the elements in the list X and not on any further choices. In [40], the author used the duality between the bases of the $\mathcal{P}$-space and the $\mathcal{D}$-space to prove a slight generalisation of the Khovanskii–Pukhlikov formula [37] that relates the volume and the number of lattice points in a smooth lattice polytope.

Our second main result is that far more general pairs of zonotopal spaces with nice properties can be constructed than the ones that were previously known. We define a new combinatorial structure called forward exchange matroid. A forward exchange matroid