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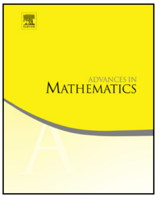

Dynamics of sound waves in an interacting Bose gas

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A B S T R A C T

We consider a non-relativistic quantum gas of N bosonic atoms confined to a box of volume Λ in physical space. The atoms interact with each other through a pair potential whose strength is inversely proportional to the density, $\rho = \frac{N}{\Lambda}$, of the gas. We study the time evolution of coherent excitations above the ground state of the gas in a regime of large volume Λ and small ratio $\frac{\Lambda}{\rho}$. The initial state of the gas is assumed to be close to a *product state* of one-particle wave functions that are approximately constant throughout the box. The initial one-particle wave function of an excitation is assumed to have a compact support independent of Λ. We derive an effective non-linear equation for the time evolution of the one-particle wave function of an excitation and establish an explicit error bound tracking the accuracy of the effective non-linear dynamics in terms of the ratio $\frac{\Lambda}{\rho}$. We conclude with a discussion of the dispersion law of low-energy excitations, recovering Bogolyubov's well-known formula for the speed of sound in the gas, and a dynamical instability for attractive two-body potentials.

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1. Introduction

In the study of the intricate dynamics of many-body systems, it is often convenient, or actually unavoidable, to resort to simpler approximate descriptions. For quantum-mechanical many-body systems of bosons it is possible to use effective one-particle equations to track the microscopic evolution of many-particle states in appropriate regimes. This tends to reduce the complexity of the problem enormously. Of course, one has to convince oneself that the approximation introduced into the analysis is not too crude but resolves the dynamical features of interest fairly accurately. To mention an example, the interaction potential exerted on a test particle in a non-linear one-particle description of the effective dynamics of a Bose gas can be chosen self-consistently as the mean potential generated by all the other particles at the position of the test particle. The mathematical analysis of such so-called *mean-field limits* goes back to work by Hepp [7] (quantum many-body systems), and by Braun and Hepp [2] and Neunzert [14] (classical many-body systems). Among other results, they have shown that the Vlasov equation effectively describes a classical many-body system while the Hartree equation describes a Bose gas in the mean-field limit. After Hepp's initial work [7] there has been a lot of effort to arrive at a mathematically rigorous understanding of quantum-mechanical mean-field limits; see [13] for an elaborate overview.

In oder to clarify the relation between our discussion and previous studies found in the existing literature, it is necessary to first explain our conventions concerning units of physical quantities and the use of dimensionless parameters:

Remark 1.1. All physical quantities appearing in this paper are made dimensionless by expressing them in terms of (dimensionful) fundamental constants of Nature or of constants characteristic of the system under consideration. In this paper, we use units in which Planck's constant and the mass of a gas atom are equal to unity. Furthermore,