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Classification of quantum groups and Lie bialgebra structures on $sl(n, \mathbb{F})$. Relations with Brauer group

Alexander Stolin, Iulia Pop *

Department of Mathematical Sciences, University of Göteborg, 41296 Göteborg, Sweden

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ABSTRACT

Given an arbitrary field $\mathbb{F}$ of characteristic 0, we study Lie bialgebra structures on $sl(n, \mathbb{F})$, based on the description of the corresponding classical double. For any Lie bialgebra structure δ, the classical double $D(sl(n, \mathbb{F}), \delta)$ is isomorphic to $sl(n, \mathbb{F}) \otimes_{\mathbb{F}} A$, where A is either $\mathbb{F}[\varepsilon]$, with $\varepsilon^2 = 0$, or $\mathbb{F} \oplus \mathbb{F}$ or a quadratic field extension of $\mathbb{F}$. In the first case, the classification leads to quasi-Frobenius Lie subalgebras of $sl(n, \mathbb{F})$. In the second and third cases, a Belavin–Drinfeld cohomology can be introduced which enables one to classify Lie bialgebras on $sl(n, \mathbb{F})$, up to gauge equivalence. The Belavin–Drinfeld untwisted and twisted cohomology sets associated to an r-matrix are computed. For the Cremmer–Gervais r-matrix in $sl(3)$, we also construct a natural map of sets between the total Belavin–Drinfeld twisted cohomology set and the Brauer group of the field $\mathbb{F}$.

* Corresponding author.
E-mail addresses: astolin@chalmers.se (A. Stolin), iulia@chalmers.se (I. Pop).

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1. Introduction

The aim of the present article is a complete classification of Lie bialgebra structures on $sl(n,\mathbb{F})$, for an arbitrary field $\mathbb{F}$ of characteristic zero. This study is motivated by our desire to classify quantum groups whose quasi-classical limit is a given simple complex Lie algebra, in our case $sl(n)$. Following [4], we recall that a quantized universal enveloping algebra (or a quantum group) over a field k of characteristic zero is a topologically free topological Hopf algebra H over the formal power series ring $k[[\hbar]]$ such that $H/\hbar H$ is isomorphic to the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ over k.

The quasi-classical limit of a quantum group is a Lie bialgebra. A Lie bialgebra is a Lie algebra $\mathfrak{g}$ together with a cobracket δ which is compatible with the Lie bracket. Given a quantum group H, with comultiplication Δ, the quasi-classical limit of H is the Lie bialgebra $\mathfrak{g}$ of primitive elements of $H/\hbar H$ and the cobracket is the restriction of the map $(\Delta - \Delta^{21})/\hbar(\mathrm{mod}\hbar)$ to $\mathfrak{g}$.

The operation of taking the semiclassical limit is a functor $SC : QUE \to LBA$ between categories of quantum groups and Lie bialgebras over k. The existence of universal quantization functors was proved by Etingof and Kazhdan [5,6]. They used Drinfeld's theory of associators to construct quantization functors for any field k of characteristic zero. More precisely, according to Theorem 2.1 in [6], if $(\mathfrak{g}, \delta)$ is a Lie bialgebra over k, then one can associate a Lie bialgebra $\mathfrak{g}_\hbar$ over $k[[\hbar]]$ defined as $(\mathfrak{g} \otimes_k k[[\hbar]], \hbar\delta)$. Moreover, there exists an equivalence $\widehat{Q}$ between the category $LBA_0(k[[\hbar]])$ of topologically free over $k[[\hbar]]$ Lie bialgebras with $\delta = 0(\mathrm{mod}\hbar)$ and the category $HA_0(k[[\hbar]])$ of topologically free Hopf algebras cocommutative modulo $\hbar$, for any $(\mathfrak{g}, \delta)$ over k.

Due to this equivalence, the classification of quantum groups whose quasi-classical limit is $\mathfrak{g}$ is equivalent to the classification of Lie bialgebra structures on $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]]$. Since any cobracket over $\mathbb{C}[[\hbar]]$ can be extended to one over $\mathbb{C}((\hbar))$ and conversely, any cobracket over $\mathbb{C}((\hbar))$, multiplied by an appropriate power of $\hbar$, can be restricted to a cobracket over $\mathbb{C}[[\hbar]]$, this in turn reduces to the problem of finding Lie bialgebras on $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((\hbar))$.

Thus, we see that the classification of quantum groups whose classical limit is $sl(n)$ is equivalent to classification of Lie bialgebra structures on $sl(n, \mathbb{C}((\hbar)))$. For this reason, we study a more general problem concerned with classification of Lie bialgebras on $sl(n,\mathbb{F})$, where $\mathbb{F}$ is a field of characteristic 0, not necessarily algebraically closed. This will include the case $\mathbb{F} = \mathbb{C}((\hbar))$.

Our classification has two important aspects. First, the Lie bialgebra structures will be classified up to gauge equivalence, in the following sense. We consider the adjoint action Ad of $GL(n)$ on $sl(n)$. Two Lie bialgebra structures δ_1 and δ_2 on $sl(n,\mathbb{F})$ are called *gauge equivalent* if there exists an element $X \in GL(n,\mathbb{F})$ such that $\delta_1(a) = (\mathrm{Ad}_X \otimes \mathrm{Ad}_X)\delta_2(\mathrm{Ad}_X^{-1}(a))$, for any $a \in sl(n,\mathbb{F})$.

As a first step towards classification, following ideas of [10], we prove that for any Lie bialgebra structure on $sl(n,\mathbb{F})$, the associated classical double is of the form $sl(n,\mathbb{F}) \otimes_{\mathbb{F}} A$,