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Derivations and Alberti representations

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ABSTRACT

We relate generalized Lebesgue decompositions of measures in terms of curve fragments ("Alberti representations") and Weaver derivations. This correspondence leads to a geometric characterization of the local norm on the Weaver cotangent bundle of a metric measure space (X, μ): the local norm of a form df "sees" how fast f grows on curve fragments "seen" by μ. This implies a new characterization of differentiability spaces in terms of the μ-a.e. equality of the local norm of df and the local Lipschitz constant of f. As a consequence, the "Lip–lip" inequality of Keith must be an equality. We also provide dimensional bounds for the module of derivations in terms of the Assouad dimension of X.

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1. Introduction

1.1. Overview

This paper studies the differentiability properties of real-valued Lipschitz functions defined on separable metric measure spaces. Two seminal works in this field are due to Cheeger [11] and Weaver [39].

In [11] Cheeger formulated a generalization of Rademacher's differentiability theorem for metric measure spaces admitting a Poincaré inequality (this is an analytic condition that has been introduced in [20] and has proven useful to generalize notions of calculus on metric measure spaces; knowing about the Poincaré inequality is *not* a prerequisite for understanding this paper); a metric measure space satisfying the *conclusion* of Cheeger's result is often called a *(Lipschitz) differentiability space* or is said to have a *(measurable/strong measurable) differentiable structure*. Applications of differentiability spaces include the study of Sobolev and quasiconformal maps in metric measure spaces [7,24] and the study of metric embeddings [12,11]. Recently Bate [8] approached this subject from a different angle by showing that differentiability spaces have a rich 1-*rectifiable structure*, which can be described in terms of Fubini-like representations of the measure which are called *Alberti representations* or 1-*rectifiable representations* [2].

Even though there are many examples of differentiability spaces, the notion of differentiable structure is rather restrictive. For example, consider $\mathbb{R}^2$ with the metric:

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|^{1/2}; \tag{1.1}$$

the existence of nowhere differentiable Hölder functions (for example the classical Weierstrass function, see [15, Exa. 11.3]) in the y-direction can be used to show that $(\mathbb{R}^2, d, \mu)$ is never a differentiability space for any choice of μ. However, for many measures, e.g. for the Lebesgue measure, there is a good notion of differentiation, or vector field, in the x-direction.

This example can be better understood using Weaver's approach [39] to differentiability, which is motivated by the study of Lipschitz algebras. This approach is, roughly speaking, based on the idea of defining *measurable vector fields* (called *derivations*) as operators acting on Lipschitz functions. Even though Weaver's approach is more flexible than Cheeger's, there are fewer works on this topic [17,34] and, apart from specific examples, it seemed unclear whether it would be possible to obtain a geometric description of derivations for a general Radon measure μ on a metric space X.

The main achievement of this paper is to provide a general approach to differentiability that can be applied to *any* Radon measure defined on a complete separable metric space; this approach unifies Weaver's theory with the study of Alberti representations and gives a geometric description of measurable vector fields and 1-forms on metric measure spaces.

Even though we build on ideas introduced in [2,8], we have to overcome significant obstacles, most notably the fact that we *do not* assume the existence of a differentiable