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New sum-product type estimates over finite fields

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Let F be a field with positive odd characteristic p. We prove a variety of new sum-product type estimates over F. They are derived from the theorem that the number of incidences between m points and n planes in the projective three-space $PG(3, F)$, with $m \geq n = O(p^2)$, is

$$O(m\sqrt{n} + km),$$

where k denotes the maximum number of collinear planes.

The main result is a significant improvement of the state-of-the-art sum-product inequality over fields with positive characteristic, namely that

$$|A \pm A| + |A \cdot A| = \Omega\left(|A|^{1+\frac{1}{5}}\right), \tag{1}$$

for any A such that $|A| < p^{\frac{5}{8}}$.

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1. Introduction

Let F be a field with positive odd characteristic p, i.e. $F = \mathbb{F}_q$, where q is a power of the prime p. In this paper we prove combinatorial-geometric estimates on sum and product sets over F, which are in a certain sense similar to those over the real and complex fields, obtained geometrically via the Szemerédi–Trotter theorem after the work of Elekes, [9]. See also [10,11,26,23,18]. Our new results appear considerably stronger that what has been known so far in the finite field setting, where the main techniques were arithmetic-combinatorial and were among other sources laid down in [5,4], see also [31] as a general reference.

For instance, we establish a new sum-product bound

$$|A \pm A| + |A \cdot A| = \Omega\left(|A|^{1+\frac{1}{5}}\right),$$

for any $A \subset F$ such that $|A| < p^{\frac{5}{8}}$. This is a considerable improvement over the previously established best results in [24,22], which were based on purely arithmetic techniques. In spirit, our main result is akin to the well-known sum-product estimate of Elekes, [9], yielding the exponent $1 + \frac{1}{4}$ for reals.

As usual, we use the notation $|\cdot|$ for cardinalities of finite sets. Symbols $\ll$, $\gg$, suppress absolute constants in inequalities, as well as respectively do the symbols O and Ω. Besides, $X = \Theta(Y)$ means that $X = O(Y)$ and $X = \Omega(Y)$. The symbols C and c stand for absolute constants, which may change from line to line. When we turn to sum-products, we use the standard notation

$$A + A = \{a_1 + a_2 : \, a_1, a_2 \in A\}$$

for the sumset $A + A$ of $A \subseteq F$, and similarly for the product set AA, alias $A \cdot A$. Sometimes we write nA for multiple sumsets, e.g. $A + A + A = 3A$, as well as $A^{-1} = \{a^{-1} : a \in A \setminus \{0\}\}$.

We use in the paper the same letter to denote a set $S \subseteq F$ and its characteristic function $S : F \to \{0, 1\}$. We write $\mathsf{E}(A, B)$ for the *additive energy* of two sets $A, B \subseteq F$, that is

$$\mathsf{E}(A, B) = |\{a_1 + b_1 = a_2 + b_2 \;:\; a_1, a_2 \in A,\, b_1, b_2 \in B\}|.$$

If $A = B$ we simply write $\mathsf{E}(A)$ instead of $\mathsf{E}(A, A)$. Similarly,

$$\mathsf{E}_k(A) = |\{a_1 - a_1' = \cdots = a_k - a_k' \;:\; a_j, a_j' \in A\}|.$$

Throughout the paper P will denote a set of m points in F^3 or $PG(3, F)$ and Π a set of n planes.

Given an arrangement $\{P, \Pi\}$ of planes and points in F^3, the set of incidences is defined as

$$I(P, \Pi) = \{(\rho, \pi) \in P \times \Pi : \, \rho \in \pi\}.$$