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Unramified degree three invariants of reductive groups ☆

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ABSTRACT

We prove that if G is a reductive group over an algebraically closed field F, then for a prime integer $p \neq \mathrm{char}(F)$, the group of unramified Galois cohomology $H^3_{\mathrm{nr}}(F(BG), \mathbb{Q}_p/\mathbb{Z}_p(2))$ is trivial for the classifying space BG of G if p is odd or the commutator subgroup of G is simple.

1. Introduction

The notion of a cohomological invariant of an algebraic group was introduced by J.-P. Serre in [18]. Let G be an algebraic group over a field F and M a Galois module over F. A *degree d invariant* of G assigns to every G-torsor over a field extension K over F an element in the Galois cohomology group $H^d(K, M)$, functorially in K. In

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this paper we consider the cohomology groups $H^d(K) = H^d(K, \mathbb{Q}/\mathbb{Z}(d-1))$, where $\mathbb{Q}/\mathbb{Z}(d-1)$ is defined as the Galois module of $(d-1)$-twisted roots of unity. The p-part of this module requires special care if $p = \operatorname{char}(F) > 0$. All degree d invariants of G form an abelian group $\operatorname{Inv}^d(G)$. An invariant is *normalized* if it takes a trivial torsor to the trivial cohomology class. The group $\operatorname{Inv}^d(G)$ is the direct sum of the subgroup $\operatorname{Inv}^d(G)_{\mathrm{norm}}$ of normalized invariants and the subgroup of *constant* invariants isomorphic to $H^d(F)$.

The group $\operatorname{Inv}^d(G)_{\mathrm{norm}}$ for small values of d is well understood. The group $\operatorname{Inv}^1(G)_{\mathrm{norm}}$ is trivial if G is connected. There is a canonical isomorphism $\operatorname{Inv}^2(G)_{\mathrm{norm}} \simeq \operatorname{Pic}(G)$ for every reductive group G (see [2, Theorem 2.4]). M. Rost proved (see [6, Part 2]) that if G is simple simply connected then the group $\operatorname{Inv}^3(G)_{\mathrm{norm}}$ is cyclic of finite order with a canonical generator called the *Rost invariant*. The group $\operatorname{Inv}^3(G)_{\mathrm{norm}}$ for an arbitrary semisimple group G was studied in [10].

For a prime integer p, write $H^d(K,p)$ and $\operatorname{Inv}^d(G,p)$ for the p-primary components of $H^d(K)$ and $\operatorname{Inv}^d(G)$ respectively. If v is a discrete valuation of a field extension K/F trivial on F with residue field $F(v)$, then there is defined the *residue* homomorphism $\partial_v : H^d(K,p) \longrightarrow H^{d-1}(F(v),p)$ for every $p \neq \operatorname{char}(F)$. An element $a \in H^d(K,p)$ is *unramified* with respect to v if $\partial_v(a) = 0$. We write $H^d_{\mathrm{nr}}(K,p)$ for the subgroup of all elements unramified with respect to every discrete valuation of K over F. An invariant in $\operatorname{Inv}^d(G,p)$ is called *unramified* if all values of the invariant over every K/F belong to $H^d_{\mathrm{nr}}(K,p)$. We write $\operatorname{Inv}^d_{\mathrm{nr}}(G,p)$ for the group of all unramified invariants.

Let V be a generically free representation of G. There are a nonempty G-invariant open subscheme $U \subset V$ and a *versal* G-torsor $U \longrightarrow X$ for a variety X over F. We think of X as an approximation of the *classifying space* BG of G. The larger the codimension of $V \setminus U$ in V the better X approximates BG. Abusing notation, we will write BG for X. Note that the stable birational type of BG is well defined.

The generic fiber of the versal G-torsor is the *generic* G-torsor over the function field $F(BG)$ of the classifying space. A theorem of Rost and Totaro asserts that the evaluation at the generic G-torsor yields an isomorphism between $\operatorname{Inv}^d(G,p)$ and the subgroup of $H^d(F(BG),p)$ of all elements unramified with respect to the discrete valuations associated with all irreducible divisors of BG. This isomorphism restricts to an isomorphism

$$\operatorname{Inv}^d_{\mathrm{nr}}(G,p) \xrightarrow{\sim} H^d_{\mathrm{nr}}(F(BG),p).$$

A classical question is whether the classifying space BG of an algebraic group G is stably rational. To disprove stable rationality of BG it suffices to show that the map $H^d(F,p) \longrightarrow H^d_{\mathrm{nr}}(F(BG),p)$ is not surjective for some d and p or, equivalently, to find a non-constant unramified invariant of G. For example, D. Saltman disproved in [14] the Noether Conjecture (that V/G is stably rational for a faithful representation V of a finite group G over an algebraically closed field) by proving that $H^2_{\mathrm{nr}}(F(BG),p) \neq H^2(F,p)$ for some G and p, i.e., by establishing a non-constant degree 2 invariant of G. E. Peyre found new examples of finite groups with non-constant unramified degree 3 invariants