

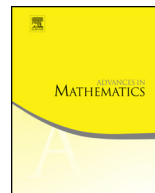


ELSEVIER

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim



Unramified degree three invariants of reductive groups [☆]



A. Merkurjev

Department of Mathematics, University of California, Los Angeles,
CA 90095-1555, USA

ARTICLE INFO

Article history:

Received 13 November 2014

Received in revised form 3 February 2016

2016

Accepted 23 February 2016

Available online 4 March 2016

Communicated by Michel Van den Bergh

MSC:

12G05

20G10

Keywords:

Reductive algebraic group

Classifying space

Unramified cohomology

ABSTRACT

We prove that if G is a reductive group over an algebraically closed field F , then for a prime integer $p \neq \text{char}(F)$, the group of unramified Galois cohomology $H_{\text{nr}}^3(F(BG), \mathbb{Q}_p/\mathbb{Z}_p(2))$ is trivial for the classifying space BG of G if p is odd or the commutator subgroup of G is simple.

© 2016 Elsevier Inc. All rights reserved.

1. Introduction

The notion of a cohomological invariant of an algebraic group was introduced by J.-P. Serre in [18]. Let G be an algebraic group over a field F and M a Galois module over F . A *degree d invariant* of G assigns to every G -torsor over a field extension K over F an element in the Galois cohomology group $H^d(K, M)$, functorially in K . In

[☆] The work has been supported by the NSF grant DMS #1160206 and the Guggenheim Fellowship.
E-mail address: merkurev@math.ucla.edu.

this paper we consider the cohomology groups $H^d(K) = H^d(K, \mathbb{Q}/\mathbb{Z}(d - 1))$, where $\mathbb{Q}/\mathbb{Z}(d - 1)$ is defined as the Galois module of $(d - 1)$ -twisted roots of unity. The p -part of this module requires special care if $p = \text{char}(F) > 0$. All degree d invariants of G form an abelian group $\text{Inv}^d(G)$. An invariant is *normalized* if it takes a trivial torsor to the trivial cohomology class. The group $\text{Inv}^d(G)$ is the direct sum of the subgroup $\text{Inv}^d(G)_{\text{norm}}$ of normalized invariants and the subgroup of *constant* invariants isomorphic to $H^d(F)$.

The group $\text{Inv}^d(G)_{\text{norm}}$ for small values of d is well understood. The group $\text{Inv}^1(G)_{\text{norm}}$ is trivial if G is connected. There is a canonical isomorphism $\text{Inv}^2(G)_{\text{norm}} \simeq \text{Pic}(G)$ for every reductive group G (see [2, Theorem 2.4]). M. Rost proved (see [6, Part 2]) that if G is simple simply connected then the group $\text{Inv}^3(G)_{\text{norm}}$ is cyclic of finite order with a canonical generator called the *Rost invariant*. The group $\text{Inv}^3(G)_{\text{norm}}$ for an arbitrary semisimple group G was studied in [10].

For a prime integer p , write $H^d(K, p)$ and $\text{Inv}^d(G, p)$ for the p -primary components of $H^d(K)$ and $\text{Inv}^d(G)$ respectively. If v is a discrete valuation of a field extension K/F trivial on F with residue field $F(v)$, then there is defined the *residue* homomorphism $\partial_v : H^d(K, p) \rightarrow H^{d-1}(F(v), p)$ for every $p \neq \text{char}(F)$. An element $a \in H^d(K, p)$ is *unramified* with respect to v if $\partial_v(a) = 0$. We write $H_{\text{nr}}^d(K, p)$ for the subgroup of all elements unramified with respect to every discrete valuation of K over F . An invariant in $\text{Inv}^d(G, p)$ is called *unramified* if all values of the invariant over every K/F belong to $H_{\text{nr}}^d(K, p)$. We write $\text{Inv}_{\text{nr}}^d(G, p)$ for the group of all unramified invariants.

Let V be a generically free representation of G . There are a nonempty G -invariant open subscheme $U \subset V$ and a *versal* G -torsor $U \rightarrow X$ for a variety X over F . We think of X as an approximation of the *classifying space* BG of G . The larger the codimension of $V \setminus U$ in V the better X approximates BG . Abusing notation, we will write BG for X . Note that the stable birational type of BG is well defined.

The generic fiber of the versal G -torsor is the *generic* G -torsor over the function field $F(BG)$ of the classifying space. A theorem of Rost and Totaro asserts that the evaluation at the generic G -torsor yields an isomorphism between $\text{Inv}^d(G, p)$ and the subgroup of $H^d(F(BG), p)$ of all elements unramified with respect to the discrete valuations associated with all irreducible divisors of BG . This isomorphism restricts to an isomorphism

$$\text{Inv}_{\text{nr}}^d(G, p) \xrightarrow{\sim} H_{\text{nr}}^d(F(BG), p).$$

A classical question is whether the classifying space BG of an algebraic group G is stably rational. To disprove stable rationality of BG it suffices to show that the map $H^d(F, p) \rightarrow H_{\text{nr}}^d(F(BG), p)$ is not surjective for some d and p or, equivalently, to find a non-constant unramified invariant of G . For example, D. Saltman disproved in [14] the Noether Conjecture (that V/G is stably rational for a faithful representation V of a finite group G over an algebraically closed field) by proving that $H_{\text{nr}}^2(F(BG), p) \neq H^2(F, p)$ for some G and p , i.e., by establishing a non-constant degree 2 invariant of G . E. Peyre found new examples of finite groups with non-constant unramified degree 3 invariants

Download English Version:

<https://daneshyari.com/en/article/4665205>

Download Persian Version:

<https://daneshyari.com/article/4665205>

[Daneshyari.com](https://daneshyari.com)