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The signature of a rough path: Uniqueness

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In the context of controlled differential equations, the signature is the exponential function on paths. B. Hambly and T. Lyons proved that the signature of a bounded variation path is trivial if and only if the path is tree-like. We extend Hambly–Lyons' result and their notion of tree-like paths to the setting of weakly geometric rough paths in a Banach space. At the heart of our approach is a new definition for reduced path and a lemma identifying the reduced path group with the space of signatures.

1. Introduction

In K.T. Chen's work [8] on the cohomology of the loop space, he defined and systematically studied the formal series of iterated integrals

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$$S(x) = 1 + \sum_{i_1} \int_0^T \mathrm{d}x_{t_1}^{i_1} X_{i_1} + \sum_{i_1,i_2} \int_0^T \int_0^{t_2} \mathrm{d}x_{t_1}^{i_1} \mathrm{d}x_{t_2}^{i_2} X_{i_1} \cdot X_{i_2} + \dots \tag{1.1}$$

where $x : [0,T] \to \mathbb{R}^d$ is a path with bounded variation and $X_1, \dots, X_d$ are formal non-commutative indeterminates. After proving a homomorphism property of the map S ([8], see (2.1) below), he gave an argument [10] that the map S restricted to appropriate classes of paths is, up to translation and reparametrisation, injective. Hambly and Lyons [17], motivated by the application of the map S in rough path theory, posed the following problem:

How to characterise the kernel of the map S?

Hambly and Lyons [17] proved that for a bounded variation path x, $S(x) = 1$ if and only if x is *tree-like*. They conjectured that the result extends to weakly geometric rough paths, a fundamental class of control paths for which controlled differential equations can be defined. Their result directly implies that the space of bounded variation paths, quotiented by the space of tree-like paths, forms a group with respect to the concatenation operation. They called this quotient space the *reduced path group*.

In [18], LeJan and Qian answered a special case of Hambly–Lyons conjecture. They proved that, when restricted on the complement of a Wiener measure zero set, the map S (defined using Stratonovich integration) is injective. There has been a number of other partial results for particular cases of weakly geometric rough paths [15,4,3,2]. A key observation in the proof of Hambly–Lyons and further refined by LeJan and Qian is that the iterated integrals of 1-forms along a path is a linear functional of the signature of the path. It turns out that a subtle variant of this idea, by considering the 1-form along the iterated integrals of the path work to prove Hambly–Lyons' conjecture.

To formulate the extension of Hambly–Lyons result, we must find the correct notion of tree-like weakly geometric rough paths. Hambly–Lyons' definition of tree-like path is inappropriate in the setting of weakly geometric rough paths. In fact, it is easy to prove that if x is an injective path with finite p-variation for some $p > 1$ and infinite 1-variation, then $x \star \overleftarrow{x}$ is in the kernel of S but isn't tree-like in the sense of Hambly–Lyons. A tree-like path x (in the sense of Hambly–Lyons) has the property that there exists a continuous function $h : [0,1] \to \mathbb{R}$, $h_t \geq 0$ for all $t \in [0,1]$, $h_1 = h_0$ and

$$h_s = h_t = \inf_{s \leq u \leq t} h_u \Longrightarrow x_s = x_t.$$

We will take an equivalent formulation of this property (see [12,16]) as the definition of tree-like path, as follows.

Definition 1.1. Let E be a topological space. A continuous path $x : [0,T] \to E$ is *tree-like* if there exists a $\mathbb{R}$-tree τ, a continuous map $\phi : [0,T] \to \tau$ and a map $\psi : \tau \to E$ such that $\phi(0) = \phi(T)$ and $x = \psi \circ \phi$.