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Enriched categories as a free cocompletion ☆

Richard Garner [a], Michael Shulman [b,1]

[a] Department of Computing, Macquarie University, North Ryde, NSW 2109, Australia
[b] Department of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA

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ABSTRACT

This paper has two objectives. The first is to develop the theory of bicategories enriched in a monoidal bicategory—categorifying the classical theory of categories enriched in a monoidal category—up to a description of the free cocompletion of an enriched bicategory under a class of weighted bicolimits. The second objective is to describe a universal property of the process assigning to a monoidal category $\mathcal{V}$ the equipment of $\mathcal{V}$-enriched categories, functors, transformations, and modules; we do so by considering, more generally, the assignation sending an equipment $\mathcal{C}$ to the equipment of $\mathcal{C}$-enriched categories, functors, transformations, and modules, and exhibiting this as the free cocompletion of a certain kind of enriched bicategory under a certain class of weighted bicolimits.

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E-mail addresses: richard.garner@mq.edu.au (R. Garner), mshulman@ias.edu, shulman@sandiego.edu (M. Shulman).

[1] Current affiliation: Department of Mathematics and Computer Science, University of San Diego, San Diego, CA 92110, USA.

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1. Introduction

The classical theory of categories enriched in a monoidal category [18] has many applications throughout mathematics. The more general notion of a category enriched in a bicategory is less well-known, but it allows one to capture also *internal* categories and *indexed* categories through enrichment, and has been used in the study of sheaves and stacks [38,3,36]. More generally still, we can enrich categories in a double category or a proarrow equipment [39,26]; the advantage of this over bicategory-enrichment is a better notion of enriched functor (see [31,9] for some examples).

In this paper we do two things:

(1) We categorify the theory of enriched categories to a theory of *bicategories* enriched in a monoidal bicategory, or more generally in a tricategory.
(2) We show that the construction "$\mathcal{C} \mapsto$ categories enriched in $\mathcal{C}$", for a bicategory or equipment $\mathcal{C}$, has a *universal property*.

While these objectives are perhaps seemingly unrelated, in fact the former is necessary for the latter: the universal property of enriched categories is expressed as a free cocompletion of a certain kind of enriched bicategory. This can be regarded as an instance of what Baez and Dolan [1] term the *microcosm principle*: the proper context in which to consider the theory of enriched categories is a categorified version of itself.

We now discuss (1) and (2) separately in somewhat more detail, beginning with (1).