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A homological representation formula of colored Alexander invariants

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We give a formula of the colored Alexander invariant in terms of the homological representation of the braid groups which we call truncated Lawrence's representation. This formula generalizes the famous Burau representation formula of the Alexander polynomial.

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1. Introduction

The Alexander polynomial is one of the most important and fundamental knot invariant having various definitions and interpretations. Each definition brings a different prospect and often leads to different generalizations.

In this paper we explore one of the generalizations of the Alexander polynomial, the *colored Alexander invariant* introduced in [1]. This is a family of invariants Φ_K^N of a link K indexed by positive integers $N = 2, 3, \ldots$, and Φ_K^2 coincides with the (multivariable) Alexander polynomial [12]. The first definition of the colored Alexander invariant in [1] uses a state-sum inspired from solvable vertex model in physics. As is already noted in [1] and clarified in [13], the N-th colored Alexander invariant Φ_K^N is a version of a quantum $\mathfrak{sl}_2$ invariant at $2N$-th root of unity. A remarkable feature of the colored Alexander invariant is that for fixed N, Φ_K^N is computed in polynomial time with respect to the number of the crossing of K, like the Alexander polynomial. This makes a sharp contrast with other quantum invariants, like the colored Jones polynomials whose computational complexities are exponential.

Throughout the paper we treat the case that K is a knot. Then the colored Alexander invariant $\Phi_K^N(\lambda)$ is a function of one variable λ. We give a new formulation of the colored Alexander invariant, in the same spirit as [6] where we gave a topological formulation of the loop expansion of the colored Jones polynomials.

In Theorem 5.3 we show that the colored Alexander invariant is written as a sum of the traces of *homological* representations which we call *truncated Lawrence's representation*. These are variants of Lawrence's representation studied in [11,5,6], derived from an action on the configuration space. Our formula can be seen as a generalization of the Burau representation formula of the Alexander polynomial and has more topological flavor.

A key result is Theorem 4.2, where we show that truncated Lawrence's representation is identified with a certain quantum $\mathfrak{sl}_2$ braid group representation, defined for the case the quantum parameter q is put as $2N$-th root of unity. This generalizes a relation between Lawrence's representation and generic quantum $\mathfrak{sl}_2$-representation [9,5], and is interesting in its own right.

Unfortunately, unlike the Burau representation formula, our formula is not completely topological since it heavily depends on a particular presentation (closed braid representative) of knots, and we cannot see its topological invariance directly, although it suggests a relationship to the topology of abelian coverings of the configuration space.

We remark that the colored Alexander invariant $\Phi_K^N(\lambda)$ at $\lambda = (N-1)$ is equal to the N-colored Jones polynomial at N-th root of unity [14], which in turn, is equal to Kashaev's invariant derived from the quantum dilogarithm function [8]. Thus, our formula also brings a new prospect for Kashaev's invariant and the volume conjecture.