Advances in Mathematics 289 (2016) 250–278

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

A Koszul category of representations of finitary Lie algebras

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ARTICLE INFO

Article history:
Received 12 May 2012
Received in revised form 16 September 2015
Accepted 30 October 2015
Available online 1 December 2015
Communicated by Roman Bezrukavnikov

MSC:
17B65
17B10
16G10

Keywords:
Koszul duality
Finitary Lie algebra

ABSTRACT

We find for each simple finitary Lie algebra $\mathfrak{g}$ a category $\mathbb{T}_{\mathfrak{g}}$ of integrable modules in which the tensor product of copies of the natural and conatural modules is injective. The objects in $\mathbb{T}_{\mathfrak{g}}$ can be defined as the finite length absolute weight modules, where by absolute weight module we mean a module which is a weight module for every splitting Cartan subalgebra of $\mathfrak{g}$. The category $\mathbb{T}_{\mathfrak{g}}$ is Koszul in the sense that it is antiequivalent to the category of locally unitary finite-dimensional modules over a certain direct limit of finite-dimensional Koszul algebras. We describe these finite-dimensional algebras explicitly. We also prove an equivalence of the categories $\mathbb{T}_{o(\infty)}$ and $\mathbb{T}_{sp(\infty)}$ corresponding respectively to the orthogonal and symplectic finitary Lie algebras $o(\infty)$, $sp(\infty)$.

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[1] Partially supported by DFG grants PE 980/2-1 and PE 980/3-1 (DFG SPP1388).
[2] Partially supported by DFG grants PE 980/2-1 and PE 980/3-1 (DFG SPP1388).
[3] Partially supported by DFG grants PE 980/2-1 and PE 980/3-1 (DFG SPP1388), as well as National Science Foundation grant 0901554.

http://dx.doi.org/10.1016/j.aim.2015.10.023
0001-8708/

1. Introduction

The classical simple complex Lie algebras $sl(n)$, $o(n)$, $sp(2n)$ have several natural infinite-dimensional versions. In this paper we concentrate on the "smallest possible" such versions: the direct limit Lie algebras $sl(\infty) := \varinjlim (sl(n))_{n\in\mathbb{Z}_{>2}}$, $o(\infty) := \varinjlim (o(n))_{n\in\mathbb{Z}_{\geq 3}}$, $sp(\infty) := \varinjlim (sp(2n))_{n\in\mathbb{Z}_{\geq 2}}$. From a traditional finite-dimensional point of view, these Lie algebras are a suitable language for various stabilization phenomena, for instance stable branching laws as studied by R. Howe, E.-C. Tan and J. Willenbring [11]. The direct limit Lie algebras $sl(\infty)$, $o(\infty)$, $sp(\infty)$ admit many characterizations: for instance, they represent (up to isomorphism) the simple finitary (locally finite) complex Lie algebras [1,2]. Alternatively, these Lie algebras are the only three locally simple locally finite complex Lie algebras which admit a root decomposition [16].

Several attempts have been made to build a basic representation theory for $\mathfrak{g} = sl(\infty)$, $o(\infty)$, $sp(\infty)$. As the only simple finite-dimensional representation of $\mathfrak{g}$ is the trivial one, one has to study infinite-dimensional representations. On the other hand, it is still possible to study representations which are close analogs of finite-dimensional representations. Such a representation should certainly be integrable, i.e. it should be isomorphic to a direct sum of finite-dimensional representations when restricted to any simple finite-dimensional subalgebra.

The first phenomenon one encounters when studying integrable representations of $\mathfrak{g}$ is that they are not in general semisimple. This phenomenon has been studied in [17] and [14], but it had not previously been placed within a known more general framework for non-semisimple categories. The main purpose of the present paper is to show that the notion of Koszulity for a category of modules over a graded ring, as defined by A. Beilinson, V. Ginzburg and W. Soergel in [4], provides an excellent tool for the study of integrable representations of $\mathfrak{g} = sl(\infty)$, $o(\infty)$, $sp(\infty)$.

In this paper we introduce the category $\mathbb{T}_\mathfrak{g}$ of tensor $\mathfrak{g}$-modules. The objects of $\mathbb{T}_\mathfrak{g}$ are defined at first by the equivalent abstract conditions of Theorem 3.4. Later we show in Corollary 4.6 that the objects of $\mathbb{T}_\mathfrak{g}$ are nothing but finite length submodules of a direct sum of several copies of the tensor algebra T of the natural and conatural representations. In the finite-dimensional case, i.e. for $sl(n)$, $o(n)$, or $sp(2n)$, the appropriate tensor algebra is a cornerstone of the theory of finite-dimensional representations (Schur–Weyl duality, etc.). In the infinite-dimensional case, the tensor algebra T was studied by Penkov and K. Styrkas in [17]; nevertheless its indecomposable direct summands were not understood until now from a categorical point of view.

We prove that these indecomposable modules are precisely the indecomposable injectives in the category $\mathbb{T}_\mathfrak{g}$. Furthermore, the category $\mathbb{T}_\mathfrak{g}$ is Koszul in the following sense: $\mathbb{T}_\mathfrak{g}$ is antiequivalent to the category of locally unitary finite-dimensional modules over an algebra $\mathcal{A}_\mathfrak{g}$ which is a direct limit of finite-dimensional Koszul algebras (see Proposition 5.1 and Theorem 5.5).

Moreover, we prove in Corollary 6.4 that for $\mathfrak{g} = sl(\infty)$ the Koszul dual algebra $(\mathcal{A}_\mathfrak{g}^!)^{\mathrm{opp}}$ is isomorphic to $\mathcal{A}_\mathfrak{g}$. This together with the main result of [17] allows us to