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Algebraic and transcendental formulas for the smallest parts function

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ABSTRACT

Building on work of Hardy and Ramanujan, Rademacher proved a well-known formula for the values of the ordinary partition function $p(n)$. More recently, Bruinier and Ono obtained an algebraic formula for these values. Here we study the smallest parts function introduced by Andrews; $\mathrm{spt}(n)$ counts the number of smallest parts in the partitions of n. The generating function for $\mathrm{spt}(n)$ forms a component of a natural mock modular form of weight 3/2 whose shadow is the Dedekind eta function. Using automorphic methods (in particular the theta lift of Bruinier and Funke), we obtain an exact formula and an algebraic formula for its values. In contrast with the case of $p(n)$, the convergence of our expression is non-trivial, and requires power savings estimates for weighted sums of Kloosterman sums for a multiplier in weight 1/2. These are proved with spectral methods (following an argument of Goldfeld and Sarnak).

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1. Introduction

Let $p(n)$ denote the ordinary partition function. Hardy and Ramanujan [31] developed the circle method to prove the asymptotic formula

$$p(n) \sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4\sqrt{3}\,n}.$$

Building on their work, Rademacher [40–42] proved the famous formula

$$p(n) = \frac{2\pi}{(24n-1)^{3/4}} \sum_{c=1}^{\infty} \frac{A_c(n)}{c} I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6c}\right), \tag{1.1}$$

where I_ν is the I-Bessel function, $A_c(n)$ is the Kloosterman sum

$$A_c(n) := \sum_{\substack{d \bmod c \\ (d,c)=1}} e^{\pi i s(d,c)} e\left(-\frac{dn}{c}\right), \quad e(x) := e^{2\pi i x} \tag{1.2}$$

and $s(d,c)$ is the Dedekind sum

$$s(d,c) := \sum_{r=1}^{c-1} \frac{r}{c}\left(\frac{dr}{c} - \left\lfloor \frac{dr}{c} \right\rfloor - \frac{1}{2}\right). \tag{1.3}$$

The existence of formula (1.1) is made possible by the fact that the generating function for $p(n)$ is a modular form of weight $-1/2$, namely

$$q^{-\frac{1}{24}} \sum_{n\geq 0} p(n) q^n = \frac{1}{\eta(\tau)}, \quad q := \exp(2\pi i \tau),$$

where $\eta(\tau)$ denotes the Dedekind eta function.

There are a number of ways to prove (1.1). For example, Pribitkin [38] obtained a proof using a modified Poincaré series which represents $\eta^{-1}(\tau)$. A similar technique can be used to obtain general formulas for the coefficients of modular forms of negative weight (see e.g. Hejhal [32, Appendix D] or Zuckerman [49]). The authors [2] recovered (1.1) from Poincaré series representing a weight $5/2$ harmonic Maass form whose shadow is $\eta^{-1}(\tau)$. As pointed out by Bruinier and Ono [21], the exact formula can be recovered from the algebraic formula (1.7) stated below (this was partially carried out by Dewar and Murty [22]). The equivalence of (1.1) and (1.7) (in a more general setting) is made explicit by [7, Proposition 7].

The smallest parts function $\mathrm{spt}(n)$, introduced by Andrews in [9], counts the number of smallest parts in the partitions of n. Andrews proved that the generating function for $\mathrm{spt}(n)$ is given by