Advances in Mathematics 289 (2016) 685–724

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

# Best constants for a family of Carleson sequences

Leonid Slavin

*University of Cincinnati, United States*

A R T I C L E I N F O

*Article history:*
Received 18 July 2014
Received in revised form 13 September 2015
Accepted 3 November 2015
Available online 7 December 2015
Communicated by Charles Fefferman

*MSC:*
primary 42A05, 42B35, 49K20

*Keywords:*
Dyadic $A_2$ weight
Carleson sequence
Bellman function

A B S T R A C T

We consider a general family of Carleson sequences associated with dyadic $A_2$ weights and find sharp — or, in one case, simply best known — upper and lower bounds for their Carleson norms in terms of the $A_2$-characteristic of the weight. The results obtained make precise and significantly generalize earlier estimates by Wittwer, Vasyunin, Beznosova, and others. We also record several corollaries, one of which is a range of new characterizations of dyadic $A_2$. Particular emphasis is placed on the relationship between sharp constants and optimizing sequences of weights; in most cases explicit optimizers are constructed. Our main estimates arise as consequences of the exact expressions, or explicit bounds, for the Bellman functions for the problem, and the paper contains a measure of Bellman-function innovation.

© 2015 Elsevier Inc. All rights reserved.

## 1. Preliminaries

We will be concerned with weights on $\mathbb{R}$, i.e. locally integrable functions that are positive almost everywhere. Our weights will be assumed to belong to the dyadic Muckenhoupt class $A_2^d$ associated with a particular lattice $\mathcal{D}$, i.e., the set of all dyadic intervals on the line uniquely determined by the position and size of the root interval. If an interval

*E-mail address:* leonid.slavin@uc.edu.

http://dx.doi.org/10.1016/j.aim.2015.11.004
0001-8708/© 2015 Elsevier Inc. All rights reserved.

$I$ is fixed, $\mathcal{D}(I)$ will stand for the unique dyadic lattice on $I$ and $\mathcal{D}_n(I)$ for the set of the dyadic subintervals of $I$ of the $n$-th generation, $\mathcal{D}_n(I) = \{J : J \in \mathcal{D}(I), |J| = 2^{-n}|I|\}$.

Let $\langle w\rangle_J$ be the average of a weight $w$ over an interval $J$, $\langle w\rangle_J = \frac{1}{|J|}\int_J w$. A weight $w$ is said to belong to $A_2^d$, written $w \in A_2^d$, if

$$[w]_{A_2^d} := \sup_{J\in\mathcal{D}} \langle w\rangle_J \langle w^{-1}\rangle_J < \infty.$$

The quantity $[w]_{A_2^d}$ is referred to as the $A_2^d$-characteristic of $w$. Observe that $[w]_{A_2^d} \geqslant 1$ and $w \in A_2^d$ if and only if $w^{-1} \in A_2^d$, in which case $[w]_{A_2^d} = [w^{-1}]_{A_2^d}$. For a number $Q \geqslant 1$, the set of all $A_2^d$ weights $w$ with $[w]_{A_2^d} \leqslant Q$ will be denoted by $A_2^{d,Q}$. If $I$ is an interval and the supremum in the above definition is taken over all $J \in \mathcal{D}(I)$ instead of all $J \in \mathcal{D}$, we will write $A_2^d(I)$ and $A_2^{d,Q}(I)$, as appropriate.

A non-negative sequence $\{c_J\}_{J\in\mathcal{D}}$ is called a Carleson sequence if for all $I \in \mathcal{D}$

$$\frac{1}{|I|}\sum_{J\in\mathcal{D}(I)} c_J \leqslant C < \infty. \tag{1.1}$$

The smallest such $C$, denoted by $\|\{c_J\}\|_{\mathcal{C}}$, is called the Carleson norm of $\{c_J\}$. The significance of such sequences in analysis stems mainly from their role in the Carleson embedding theorem and related results. A key example is the following lemma whose proof can be found in [10] (in a more general, weighted setting). We will use this lemma to derive an important corollary of our main estimates.

**Lemma 1.1** *(Carleson Lemma). A sequence $\{c_J\}_{J\in\mathcal{D}}$ is a Carleson sequence with norm $B$ if and only if for all non-negative, measurable functions $F$ on the line,*

$$\sum_{J\in\mathcal{D}} c_J \inf_{x\in J} F(x) \leqslant B \int_{\mathbb{R}} F(x)\,dx.$$

This inequality suggests that it may be important to have good estimates of the Carleson norms of sequences one uses. One specific context where such need arises is when studying dyadic paraproducts in weighted settings. For example, in [1] and [10], the authors estimate the norms of paraproducts on $L^2(w)$, with $w \in A_2^d$. In this situation, the sequences of interest are often of the form

$$c_J^{(\alpha)}(w) := |J|\,\langle w\rangle_J^\alpha \langle w^{-1}\rangle_J^\alpha \left[\frac{(\Delta_J w)^2}{\langle w\rangle_J^2} + \frac{(\Delta_J w^{-1})^2}{\langle w^{-1}\rangle_J^2}\right],$$

where $\Delta_J(\cdot) = \langle\cdot\rangle_{J^-} - \langle\cdot\rangle_{J^+}$, and $J^\pm$ are the two halves of $J$. In [1], Beznosova showed that the norm of $\{c_J^{(1/4)}(w)\}$ is no greater than $C[w]_{A_2^d}^{1/4}$ for a numerical constant $C$. In [14], Nazarov and Volberg extended this estimate, proving that

$$\|\{c_J^{(\alpha)}(w)\}\|_{\mathcal{C}} \leqslant \frac{C}{\alpha - 2\alpha^2}[w]_{A_2^d}^\alpha, \quad \alpha \in (0, 1/2). \tag{1.2}$$

Download English Version:

https://daneshyari.com/en/article/4665232

Download Persian Version:

https://daneshyari.com/article/4665232

Daneshyari.com