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Self-affine manifolds

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ABSTRACT

This paper studies closed 3-manifolds which are the attractors of a system of finitely many affine contractions that tile $\mathbb{R}^3$. Such attractors are called self-affine tiles. Effective characterization and recognition theorems for these 3-manifolds as well as theoretical generalizations of these results to higher dimensions are established. The methods developed build a bridge linking geometric topology with iterated function systems and their attractors.

A method to model self-affine tiles by simple iterative systems is developed in order to study their topology. The model is functorial in the sense that there is an easily computable map that induces isomorphisms between the natural subdivisions of the attractor of the model and the self-affine tile. It has many beneficial qualities including ease of computation allowing one to determine topological properties of the attractor of the model such as connectedness and whether it is a manifold. The induced map between the attractor of the model and the self-affine tile is a quotient map and can be checked in certain cases to be monotone or cell-like. Deep theorems from geometric topology are applied to characterize and develop algorithms to recognize when a self-affine tile is a topological or generalized manifold in all dimensions. These new tools are used to check that several self-affine tiles in the literature are 3-balls. An example of a wild 3-dimensional self-affine tile is given whose boundary is a topological 2-sphere

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but which is not itself a 3-ball. The paper describes how any 3-dimensional handlebody can be given the structure of a self-affine 3-manifold. It is conjectured that every self-affine tile which is a manifold is a handlebody.

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1. Introduction

A great deal of work in the literature has concentrated on tilings of $\mathbb{R}^n$ whose tiles are defined by a finite collection of contractions. One of the most prevalent examples are tilings by *self-affine tiles* where the contractions are affine translates of a single linear contraction. A long-standing open question is whether there exists a closed 3-manifold which is a nontrivial self-affine tile, a so-called *self-affine 3-manifold*. To settle this question in the affirmative the current paper effectively characterizes and recognizes self-affine 3-manifolds and gives theoretical generalizations of these results to higher dimensions. The methods developed in this paper build a bridge linking two previously unrelated areas of mathematics: geometric topology on the one side and iterated function systems and their attractors on the other side.

Much research is devoted to how a subset of the Euclidean space can admit a tiling by self-affine tiles. In the planar case, the topology of these tiles has been studied thoroughly. Much less is known about the topology of self-affine tiles of Euclidean 3-space. In particular it has been an open question as to which (if any) 3-manifolds admit a nontrivial self-affine tiling of $\mathbb{R}^3$. A number of examples have appeared in the literature which were conjectured to be self-affine tilings of $\mathbb{R}^3$ by 3-balls. In the current paper we address these questions by describing an often effective method of determining that a given 3-dimensional self-affine tile is a tamely embedded 3-manifold. The method gives affirmative answers for the previously conjectured examples, and is also used to give examples of 3-dimensional self-affine tiles which are handlebodies of higher genus. Examples are also given of self-affine tiles in $\mathbb{R}^3$ whose boundaries are wildly embedded