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Nilprogressions and groups with moderate growth

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We show that doubling at some large scale in a Cayley graph implies uniform doubling at all subsequent scales. The proof is based on the structure theorem for approximate subgroups proved by Green, Tao and the first author. We also give a number of applications to the geometry and spectrum of finite Cayley graphs. For example, we show that a finite group has moderate growth in the sense of Diaconis and Saloff-Coste if and only if its diameter is larger than a fixed power of the cardinality of the group. We call such groups almost flat and show that they have a subgroup of bounded index admitting a cyclic quotient of comparable diameter. We also give bounds on the Cheeger constant, first eigenvalue of the Laplacian, and mixing time. This can be seen as a finite-group version of Gromov's theorem on groups with polynomial growth. It also improves on a result of Lackenby regarding property (τ) in towers of coverings. Another consequence is a universal upper bound on the diameter of all finite simple groups, independent of the CFSG.

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1. Introduction

Let G be a group generated by a finite, symmetric subset S. Here, and throughout this paper, by writing that S is *symmetric* we mean that if s belongs to S then so does s^{-1}, *and* that S contains the identity. When speaking of the growth of G with respect to S, we refer to the behaviour of the sequence of cardinalities

$$|S|, |S^2|, |S^3|, \dots,$$

where we denote by S^n the n-fold product set $\{s_1 \cdot \dots \cdot s_n; s_i \in S\}$. This is the ball of radius n in the Cayley graph of G relative to S. In the event that G is finite, we also consider the *diameter* $\mathrm{diam}_S(G)$ of G with respect to S, which is defined to be the minimum n such that $S^n = G$.

According to Gromov's polynomial growth theorem [23], if the sequence $\{|S^n|\}$ is bounded above by a polynomial function of n, then G has a finite-index nilpotent subgroup. In this paper we are concerned with some refinements of this theorem, mostly in the context of finite groups. In particular, we study the relations between the diameter and the cardinality of a finite group on the one hand, and various interesting invariants such as the Cheeger constant, the first non-zero eigenvalue of the Laplace operator, and the mixing time of the associated Cayley graph on the other.

Our two main results are Theorems 1.1 and 1.3 below. The first concerns the doubling property at one given scale in an arbitrary Cayley graph.

Theorem 1.1 *(Doubling at some scale implies doubling at all larger scales). For every $K \geq 1$ there exist $n_0 = n_0(K) \in \mathbb{N}$ and $\theta(K) \geq 1$, such that if S is a finite symmetric set inside some group, and if there exists $n \geq n_0$ for which*

$$|S^{2n+1}| \leq K|S^n|, \tag{1.1}$$

then for every $m \geq n$ and every $c \in \mathbb{N}$ we have $|S^{cm}| \leq \theta(K)^c |S^m|$.