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Green polynomials of Weyl groups, elliptic pairings, and the extended Dirac index ☆

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A B S T R A C T

In this paper, we give a uniform construction of irreducible genuine characters of the Pin cover $\widetilde{W}$ of a Weyl group W, and put them into the context of theory of Springer representations. In the process, we provide a direct connection between Springer theory, via Green polynomials, the irreducible representations of $\widetilde{W}$, and an extended Dirac operator for graded Hecke algebras. We also introduce a $\mathbf{q}$-elliptic pairing for W with respect to the reflection representation V. These constructions are of independent interest. The $\mathbf{q}$-elliptic pairing is a generalization of the elliptic pairing of W introduced by Reeder, and it is also related to S. Kato's

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notion of (graded) Kostka systems for the semidirect product $A_W = \mathbb{C}[W] \ltimes S(V)$.

1. Introduction

1.1. Graded affine Hecke algebras were defined by Lusztig [22] in his study of representations of reductive p-adic groups and Iwahori–Hecke algebras. A Dirac operator $\mathcal{D}$ for graded affine Hecke algebras was defined in [3], and, by analogy with the setting of Dirac theory for $(\mathfrak{g}, K)$-modules of real reductive groups, the notion of Dirac cohomology was introduced. The Dirac cohomology and the Dirac index in the Hecke algebra setting were further studied in [10,11]. The Dirac cohomology spaces are representations for a certain double cover ("pin cover") $\widetilde{W}$ of the Weyl group W. The irreducible representations of $\widetilde{W}$ had been classified case by case in the work of Schur, Morris, Read and others, see for example [26,28,40]. Recently, it was noticed in [9] (again case by case) that there is a close relation between the representation theory of $\widetilde{W}$ and the geometry of the nilpotent cone in semisimple Lie algebras $\mathfrak{g}$.

1.2. In this paper, we provide a direct link between:

(a) an (extended) Springer W-action on cohomology groups;
(b) the irreducible representations of $\widetilde{W}$, and
(c) an (extended) Dirac index for tempered modules of graded Hecke algebras.

Here in (a) and (c), there is a canonical $\mathbb{Z}/2\mathbb{Z}$-action, induced from the conjugation of the longest Weyl group element w_0 on W. The $\mathbb{Z}/2\mathbb{Z}$-action looks irrelevant to the $\mathbb{Z}/2\mathbb{Z}$-covering $\widetilde{W} \to W$. However, the remarkable thing is that one gets many more representations of $\widetilde{W}$ with nontrivial extended Dirac index by using the twist given by such $\mathbb{Z}/2\mathbb{Z}$-action. One of the main results (see Theorem 1.2 below) shows that all the irreducible genuine $\widetilde{W}$-representations (i.e., those that do not factor through W) appear in this way. This leads to a new, uniform construction of the irreducible genuine $\widetilde{W}$-characters.

1.3. To describe the role of this $\mathbb{Z}/2\mathbb{Z}$-action, let us first make a short digression on elliptic representation theory.

Elliptic representation theory, introduced by Arthur [1], studies the Grothendieck group of certain representations of a Lie-theoretic group modulo those induced from proper parabolic subgroups. The elliptic theory of representations of semisimple p-adic groups and Iwahori–Hecke algebras was further studied intensively, e.g., Schneider–Stuhler [32], Bezrukavnikov [6], Reeder [29], Opdam–Solleveld [27]. In [29], Reeder also described the elliptic representations of W in terms of Springer correspondence. He proved an orthogonality result of Springer representations and proved that it is exactly