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The extension algebra of some cohomological Mackey functors

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ABSTRACT

Let k be a field of characteristic p. We construct a new inflation functor for cohomological Mackey functors for finite groups over k. Using this inflation functor, we give an explicit presentation of the graded algebra of self-extensions of the simple functor S_1^G, when p is odd and G is an elementary abelian p-group.

1. Introduction

Let k be a field and G be a finite group. The theory of *Mackey functors* and *cohomological Mackey functors* for G over k originates in the work of Green [5] and Dress [4],

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at the beginning of the 70s. It can be viewed as the theory of induction and restriction, when we forget the particular framework of linear representations of G. Many important developments have been achieved since, culminating in the comprehensive and seminal paper by Thévenaz and Webb [6] in 1995, where the authors introduced *the Mackey algebra* $\mu_k(G)$, and showed, among many other fundamental results, that the category of Mackey functors for G over k is equivalent to the category of $\mu_k(G)$-modules. Similarly, they showed that the subcategory $\mathsf{M}_k^c(G)$ of cohomological Mackey functors for G over k is equivalent to the category of $co\mu_k(G)$-modules, where $co\mu_k(G)$ is a specific quotient of $\mu_k(G)$, called *the cohomological Mackey algebra*.

The algebras $\mu_k(G)$ and $co\mu_k(G)$ share many similarities with the group algebra kG: e.g., they are finite dimensional k-vector spaces, of dimension independent of k, the Maschke theorem holds, there is a good theory of decomposition from characteristic 0 to characteristic p, etc. These resemblances raise as natural question, whether a given theorem on kG will admit an analogue for $\mu_k(G)$ or $co\mu_k(G)$.

This was the main motivation in [2], where the question of complexity of cohomological Mackey functors was solved (in the only non-trivial case where k is a field of positive characteristic p dividing the order of G). This question amounts to computing all extension groups between simple cohomological Mackey functors for G. It was shown in [2] that one can assume that G is a p-group, and in this case these extension groups can be determined from the knowledge of sufficient information on the algebra $\mathcal{E} = \mathrm{Ext}^*(S_{\mathbf{1}}^E, S_{\mathbf{1}}^E)$ of self-extensions of a particular simple functor $S_{\mathbf{1}}^E$ for some subquotients E of G. Along the way, a presentation of this algebra when E is elementary abelian and $p = 2$ was given, together with a formula for the Poincaré series. In the case $p > 2$, no such presentation was given, and a conjecture was proposed for the Poincaré series of $\mathcal{E}$. In the same article, the conjecture was proved in the case $p = 3$.

This paper settles completely the case $p > 2$: a presentation of $\mathcal{E}$ is given, and, as a corollary, the forementioned conjecture is proved. The main results are the following, where $S_{1,W}^H$ denotes the simple functor for the group H defined as in 2.15. To simplify notation, when $W = k$ is the trivial module, we drop this subscript. We start by the construction of a new inflation functor for cohomological Mackey functors:

1.1. Theorem. *Let k be a field, let G be a finite group, let $N \trianglelefteq G$ and let V be a simple $k(G/N)$-module. Then there exists an exact functor $\sigma_{G/N}^G$ from $\mathsf{M}_k^c(G/N)$ to $\mathsf{M}_k^c(G)$ satisfying*

$$\sigma_{G/N}^G(S_{\mathbf{1},V}^{G/N}) = S_{\mathbf{1},\mathrm{Inf}_{G/N}^G V}^G \quad .$$

Suppose, moreover, that $G = N \rtimes H$ is the semidirect product of N by a group H.

1. *If V is a kH-module, let $\tilde{V}$ be the kG-module $\mathrm{Inf}_{G/N}^G \mathrm{Iso}_H^{G/N} V$. Then the restriction of $\tilde{V}$ to H is isomorphic to V.*
2. *The composition $\mathrm{Res}_H^G \sigma_{G/N}^G \mathrm{Iso}_H^{G/N}$ is isomorphic to the identity functor of $\mathsf{M}_k^c(H)$.*