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Overconvergent generalised eigenforms of weight one and class fields of real quadratic fields

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ABSTRACT

This article examines the Fourier expansions of certain non-classical p-adic modular forms of weight one: the eponymous *generalised eigenforms* of the title, so called because they lie in a generalised eigenspace for the Hecke operators. When this generalised eigenspace contains the theta series attached to a character of a real quadratic field K in which the prime p splits, the coefficients of the attendant generalised eigenform are expressed as p-adic logarithms of algebraic numbers belonging to an idoneous ring class field of K. This suggests an approach to "explicit class field theory" for real quadratic fields which is simpler than the one based on Stark's conjecture or its p-adic variants, and is perhaps closer in spirit to the classical theory of singular moduli.

1. Introduction and statement of the main result

Fourier coefficients of modular forms often describe interesting arithmetic functions. Classical examples are the partition function, the divisor function, and representation

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numbers of quadratic forms, which are related to Fourier coefficients of the Dedekind eta function, Eisenstein series, and theta series, respectively. A more modern instance is the appearance of Frobenius traces of two-dimensional Galois representations as Fourier coefficients of normalised eigenforms. Just as germane to an understanding of the present work are the topological and arithmetic intersection numbers of special cycles arising in the formulae of Hirzebruch–Zagier, Gross–Kohnen–Zagier, and their vast generalisations growing out of the "Kudla program".

The main theorem of this paper expresses the Fourier coefficients of certain p-adic modular forms of weight one as p-adic logarithms of algebraic numbers in ring class fields of real quadratic fields. It suggests an approach to "explicit class field theory" for real quadratic fields which is simpler than the one based on Stark's (still unproved) conjecture [17] or Gross's (more tractable) p-adic variant [13]. An analogy with the growing body of work on Fourier coefficients of incoherent Eisenstein series and weak harmonic Maass forms suggests that this approach is perhaps closer in spirit to the classical theory of singular moduli.

To set the stage for the main result, let K be a real quadratic field of discriminant $D > 0$ and let χ_K denote the even quadratic Dirichlet character associated to it. Let

$$\psi : G_K := \mathrm{Gal}(\bar{K}/K) \longrightarrow \mathbb{C}^\times$$

be a ray class character (of order m, conductor $\mathfrak{f}_\psi$ and central character χ_ψ) which is of *mixed signature*, i.e., which is even at precisely one of the infinite places of K and odd at the other. Hecke's theta series $g := \theta_\psi$ attached to ψ is a holomorphic newform of weight one, level N and nebentype character χ with Fourier coefficients in $L := \mathbb{Q}(\mu_m)$, where

$$N = D \cdot \mathrm{Norm}_{K/\mathbb{Q}}\, \mathfrak{f}_\psi, \qquad \chi = \chi_K \chi_\psi.$$

Fix a prime $p = \mathfrak{p}\mathfrak{p}'$ which does not divide N and is split in $K/\mathbb{Q}$. The eigenform g is said to be *regular* at p if the Hecke polynomial

$$x^2 - a_p(g)x + \chi(p) = (x - \psi(\mathfrak{p}'))(x - \psi(\mathfrak{p})) =: (x - \alpha)(x - \beta)$$

has distinct roots. Assume henceforth that this regularity hypothesis holds, and let

$$g_\alpha(z) := g(z) - \beta g(pz), \qquad g_\beta(z) := g(z) - \alpha g(pz)$$

be the two distinct p-stabilisations of g, which are eigenvectors for the U_p operator with eigenvalues α and β respectively. Note that these stabilisations are both ordinary, since α and β are roots of unity.

Let $S_k(N, \chi)$ (resp. $S_k^{(p)}(N, \chi)$) denote the space of classical (resp. p-adic overconvergent) modular forms of weight k, level N and character χ, with coefficients in $\mathbb{C}_p$. The Hecke algebra $\mathbb{T}$ of level Np generated over $\mathbb{Q}$ by the operators T_ℓ with $\ell \nmid Np$ and U_q with