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Unique continuation type theorem for the self-similar Euler equations

Dongho Chae

Chung-Ang University, Department of Mathematics, Seoul 156-756, Republic of Korea

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ABSTRACT

We prove a unique continuation type theorem for the self-similar Euler equations in $\mathbb{R}^3$, assuming the time periodicity. Namely, if a time periodic solution $V(y,s)$ of the time dependent self-similar Euler equations has the property that $V(0,s)=0$ for all $s\in[0,S_0]$, where S_0 is the time period, and $y=0$ is a local extremal point of $V(y,s)$ near the origin, then, $V(y,s)=0$ for all $(y,s)\in\mathbb{R}^3\times[0,S_0]$. A similar result holds for more general system with arbitrary coefficients, and also for the inviscid incompressible magnetohydrodynamic (MHD) system. As a consequence we obtain new criteria for the absence of the discretely self-similar singularities for the 3D Euler equations and the inviscid 3D MHD.

1. The main theorems

1.1. The Euler equations

We are concerned on the Cauchy problem for the incompressible 3D Euler equations in $\mathbb{R}^3$:

E-mail address: dchae@cau.ac.kr.

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$$(E)\begin{cases} \dfrac{\partial v}{\partial t}+v\cdot\nabla v=-\nabla p,\\ \mathrm{div}\, v=0,\\ v(x,0)=v_0(x),\end{cases}$$

where $v(x,t)=(v_1(x,t),v_2(x,t),v_3(x,t))$ is the velocity, $p=p(x,t)$ is the pressure, and $v_0(x)$ is the initial data satisfying $\mathrm{div} v_0=0$. For (E) it is well-known that for the initial data belonging to the standard Sobolev space, $v_0\in H^m(\mathbb{R}^3)$, $m>5/2$, the local well-posedness holds (see e.g. [15]). The question of the finite time singularity for the local classical solution, however, is still an outstanding open problem (see [2,11]). For survey books or articles on the study of the finite time blow-up problem of (E) we refer e.g. [17,10,1]. For studies of the possibility of self-similar blow-up or its generalized version, discretely self-similar blow-up for the Euler and the Navier–Stokes equations there are previous studies (e.g. [19,4,6,5,7–9,21,22,18]). For studies of self-similar solutions to the other nonlinear equations we refer e.g. [13,14,12].

Once the velocity $v(x,t)$ is known for a solution to (E), the pressure is determined by solving the Poisson equation $\Delta p=-\mathrm{div}\,\mathrm{div}(v\otimes v)$, and therefore, up to an addition of harmonic function, is given by the well-known formula

$$p(x,t)=-\frac{|v(x,t)|^2}{3}+\frac{1}{4\pi}P.V.\int_{\mathbb{R}^3}\frac{3(y\cdot v(y,t))^2-|y|^2||v(y,t)|^2}{|y|^5}dy. \tag{1.1}$$

Given a solution (v,p) of (E) and $(x_*,T)\in\mathbb{R}^3\times\mathbb{R}_+$, we consider a canonical transform of $(v,p)\mapsto(V,P)$, called the self-similar transform, which is defined by

$$v(x,t)=\frac{1}{(T-t)^{\frac{\alpha}{\alpha+1}}}V(y,s),\quad p(x,t)=\frac{1}{(T-t)^{\frac{2\alpha}{\alpha+1}}}P(y,s), \tag{1.2}$$

where

$$y=\frac{x-x_*}{(T-t)^{\frac{1}{\alpha+1}}},\quad s=\log\left(\frac{T}{T-t}\right), \tag{1.3}$$

and $\alpha\neq-1$. Note that our physical space–time domain $\mathbb{R}^3\times[0,T)$ is transformed into $\mathbb{R}^3\times[0,\infty)$. Substituting (1.2)–(1.3) into (E), we obtain the following system for (V,P):

$$(\mathrm{SSE})_\alpha\begin{cases}\dfrac{\partial V}{\partial s}+\dfrac{\alpha}{\alpha+1}V+\dfrac{1}{\alpha+1}(y\cdot\nabla)V+(V\cdot\nabla)V=-\nabla P,\\ \mathrm{div}\, V=0,\\ V(y,0)=V_0(y)=T^{\frac{\alpha}{\alpha+1}}v_0(T^{\frac{1}{\alpha+1}}y).\end{cases} \tag{1.4}$$

Let us temporarily denote the pressure P of (1.4) by $\overline{P}$. Taking operation div on the first equation of (1.4) we obtain the Poisson equation $\Delta\overline{P}=-\mathrm{div}\,\mathrm{div}(V\otimes V)$. Since this equation determines (up to an addition of harmonic function) the self-similar pressure $\overline{P}$ by the self-similar velocity V, simultaneous prescription of V, P by (1.2) may cause the